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Some Types of Irregular Labeling of Diamond Networks on Ten Vertices

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Abstract

There are three interesting parameters in irregular networks based on total labelling, i.e. the total vertex irregularity strength, the total edge irregularity strength, and the total irregularity strength of a graph. Besides that, there is a parameter based on edge labelling, i.e., the irregular labelling. In this paper, we determined the four parameters for diamond graph on eight vertices.

Keywords: Diamond graph, irregular labeling, network

1. INTRODUCTION

Many types of labelling graphs that appeared after Sedlàček introduced concept labelling graphs in 1963 [12]. One of all is irregular labelling introduced by Chartrand, et al. in 1988 [4]. Let G be a simple graph. A function $\gamma\colon E \to \{1,2,3,\dots,b\}$ is referred to as an edge irregular b-labelling if all vertices of G have vary weights. The weight of vertex u is $\sum \gamma(e)$ for an edge e incident to u. The irregularity strength of G, given a symbol $s(G)$, is the minimum positive integer b so that G has an irregular b- labeling.

Related to the irregularity strength of some graph, Nurdin, et al. decided the irregularity strength of caterpillar graph [10]. Aigner and Triesch found that $s(G) \le n-1$ for the number of vertices of G (different from a triangle) is n [2]. In fact Faudree and Lehel proved that $\lceil n+d-1)/d\rceil \le s(G) \le \left\lceil \frac{n}{2} \right\rceil + 9$ for G is d −regular ($d \ge 2$) graph on n vertices [5].

Motivated by the notion of irregular labeling and the total labeling, Bača, et al. [3] presented the total vertex irregular labeling and the total edge irregular labeling of graphs. A function $\beta\colon V \cup E \to \{1,2,3,\dots,n\}$ is referred an edge irregular total n-labeling if, the weight for all edges are different. The weight of edge uv under the total labeling β is $\beta(u) + \beta(uv) + \beta(v)$. The total edge irregularity strength of G, denoted by $tes(G)$, is the minimum positive integer n so that G has a total edge irregular n-labeling.

Related to the total edge irregularity strength of some graph, Bača, et al. [3] determined the total edge irregularity strength of certain classes of simple graphs, such as paths, cycles, stars, and

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wheels. The total edge irregularity strength of any graphs constructed from stars found by Nurdin and H.K. Kim [9].

Besides the total edge irregular labeling, Bača, et al. [3] also introduced the total vertex irregular labeling. For G is a simple graph, a function $\mu: V \cup E \rightarrow \{1, 2, 3, \dots, m\}$ is referred a vertex irregular total m-labeling if, the weight of all vertices are different. The weight of vertex u is $\mu(u) + \sum \mu(e)$ for an edge e is edge incident to u. The total vertex irregularity strength of G, indicated by $\mathrm{tvs}(G)$, is the minimum natural number m such that G has a total vertex irregular m-labeling.

Related to the total vertex irregularity strengths for various classes of graphs have been decided. For instance, Wijaya and Slamin determined the total vertex irregularity strength of friendship, fans, suns, and wheels graphs [15]. In 2013, Siddiqui, et. al. decided the total vertex irregularity strength of dijoint union of helm graph [14]. Ahmad, et al. decided the total edge and vertex irregularity strength of generalized Halin graphs [1].

In 2013, Marzuki, Salman, and Miller presented the totally irregular total on a simple graph G. A function $\theta: V \cup E \rightarrow \{1, 2, 3, \dots, t\}$ is referred a totally irregular total t $-$labeling if the weight of all edges and the weight of each vertex are different. The total irregularity strength of G, indicated by $\mathrm{ts}(G)$, is the minimum natural number t so that G has totally irregular total t $-$labeling [7].

Associated with the total irregularity strength, Salman and Ramdani provide the total irregularity strength of the Cartesian product of P_2 and a cycle, a star, a fan, and a path graph [11]. Marzuki, et al. decided total irregularity strength of cycle and path [7].

Diamond Graph

A diamond graph is constructed from ladder graph as define below.

Definition 1. [6] *A ladder graph L_n for $n \geq 2$ is a connected graph of order $2n$, constructed from $P_2 \times P_n$ with the vertex set*

$$V(L_n) = \{u_i, v_i: 1 \leq i \leq n\}$$

and the edge set

$$E(L_n) = \{u_i u_{i+1}, v_i v_{i+1}: 1 \leq i \leq n-1\} \cup \{u_i v_i: 1 \leq i \leq n\}.$$

Definition 2. [14] *Let L_n be a ladder graph as defined in Definition 1. A triangular ladder graph TL_n for $n \geq 2$ is a connected graph of order $2n$, constructed from L_n by adding the edge $u_i v_{i+1}$ for $1 \leq i \leq n-1$. A triangular ladder graph of order $2n-1$ can be obtained by removing a vertex of degree 2 of TL_n of order $2n$.*

Definition 3. [6] *A diamond graph, denoted by Br_n, with $n \geq 2$, is a connected graph of order $2n$, constructed from the triangular ladder graph TL_n of order $2n-1$ by adding a vertex x and n edges such that x and v_i (in Definition 1.) are adjacent.*

The vertex set of Br_n is

$$V(\mathrm{Br}_n) = \{x\} \cup \{v_i | i = 1,2, \dots, n\} \cup \{u_i | i = 1,2, \dots, n-1\},$$

and the edge set of Br_n is

$$E(\mathrm{Br}_n) = \{x v_i | i = 1,2, \dots, n\} \cup \{v_i v_{i+1} | i = 1,2, \dots, n-1\}$$

$$\cup\ \{u_i u_{i+1} | i = 1,2, \dots, n-2\} \cup \{u_i v_i | i = 1,2, \dots, n-1\}$$

$$\cup \{u_i v_{i+1} | i = 1,2, \dots, n-1\}.$$

An illustration of Definition 3 is shown in Figure 1.

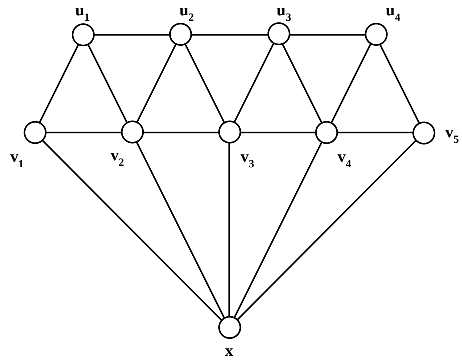

Figure 1. A Diamond Graph Br_5

2. MAIN RESULTS

1. The Irregular Labeling of Diamond Graph

In this subsection, we give the found the irregularity strength of diamond graph Br_5 as follows.

***Theorem* 1.** *The irregularity strength of diamond graph Br_5 is* 3.

Proof. Note that the number edges of Br_5 is 20, the sequences of degree of vertices are 3, 4, and 5. Since the weight of a vertex is the sum of label of every edges incident to the vertex, the largest weight a vertex of Br_5 more than 13. Such that, $s(Br_5) \geq \max\left\{\left\lceil\frac{13}{3}\right\rceil, \left\lceil\frac{13}{4}\right\rceil, \left\lceil\frac{13}{5}\right\rceil\right\} = 3$. To prove $s(Br_5) \leq 3$, we have to construct an irregular 3-labeling of diamond graph Br_5 as follows. Define an edge labelling γ on Br_5 as follows:

$$\gamma(xv_i) = \begin{cases} 1 & \text{for } i = 1,5 \\ 2 & \text{for } i = 2,4 \\ 3 & \text{for } i = 3, \end{cases}$$

$$\gamma(v_i v_{i+1}) = \begin{cases} 1 & \text{for } i = 1 \\ 3 & \text{for } i = 2,3 \\ 2 & \text{for } i = 4, \end{cases}$$

$$\gamma(v_i u_j) = \begin{cases} 1 & \text{for } (i,j) = (1,1),(5,4) \\ 2 & \text{for } (i,j) = (2,1),(2,2),(3,2),(3,3),(4,3),(4,4), \end{cases}$$

$$\gamma(u_i u_{i+1}) = \begin{cases} 2 & \text{for } i = 1 \\ 1 & \text{for } i = 2 \\ 3 & \text{for } i = 3. \end{cases}$$

By this definition we can find that the weight of all vertices are different and γ is defining a function $\gamma: E \to \{1, 2, 3\}$. Such that, γ is defining an irregular $3-$labeling of Br_5. So that, $s(Br_5) \leq 3$. ■

2. The Total Vertex Irregular Labeling of Diamond Graph

Nurdin, et al. provide the lower bound of the total vertex irregularity strength of any graph as in Theorem 2.

***Theorem* 2.** [8] *Let G be a graph have n_i vertices of degree i for $i = \delta, \delta + 1, \delta + 2, \cdots, \Delta$ with δ and Δ are smallest and biggest degree of G, respectively. Then*

$$tvs(G) \geq \max\left\{\left\lceil\frac{\delta + n_\delta}{\delta + 1}\right\rceil, \left\lceil\frac{\delta + n_\delta + n_{\delta+1}}{\delta + 2}\right\rceil, \cdots, \left\lceil\frac{\delta + \sum_{i=\delta}^{\Delta} n_i}{\Delta + 1}\right\rceil\right\}.$$

Besides that, Bača, et al provide the lower and upper bound of the total vertex irregularity strength of any graph, as follows

***Theorem* 3.** [3] *Let G be a (p, q) graph with smallest degree δ and biggest degree Δ, then* $\left\lceil\frac{p+\delta}{\Delta+1}\right\rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1$.

In Theorem 4, we give the value exact of the total vertex irregularity strength of diamond graph Br_5.

***Theorem* 4.** *The total vertex irregularity strength of diamond graph Br_5 is* 3.

Proof. Since the number vertices of degree 3 of Br_5 is 4, the number vertices of degree 4 is 2, and the number vertices of degree five is 4, according to Theorem 1 or Theorem 2, we found that the total vertex irregularity strength of diamond graph Br_5 bigger or equal to $\max\left\{\left\lceil\frac{7}{4}\right\rceil, \left\lceil\frac{9}{5}\right\rceil, \left\lceil\frac{13}{6}\right\rceil\right\} = 3$. To find the upper bound of the total vertex irregularity strength of diamond graph Br_5 we have to construct a total vertex irregular 3-labeling of diamond graph Br_5 as follows. Define a total labelling μ on Br_5 as follows:

$$\mu(xv_i) = \begin{cases} 2 & \text{for } i = 1,3 \\ 1 & \text{for } i = 2,5 \\ 3 & \text{for } i = 4, \end{cases}$$

$$\mu(v_i v_{i+1}) = \begin{cases} 3 & \text{for } i = 1,3 \\ 2 & \text{for } i = 2,4, \end{cases}$$

$$\mu(v_i u_j) = \begin{cases} 2 & \text{for } (i,j) = (1,1), (2,1), (3,2), (3,3) \\ 1 & \text{for } (i,j) = (2,2), (4,3), (4,4), (5,4), \end{cases}$$

$$\mu(u_i u_{i+1}) = \begin{cases} 2 & \text{for } i = 1 \\ 1 & \text{for } i = 2,3, \end{cases}$$

$$\mu(x) = 1,$$

$$\mu(v_i) = \begin{cases} 2 & \text{for } 1 \leq i \leq 4 \\ 1 & \text{for } i = 5, \end{cases}$$

$$\mu(u_i) = \begin{cases} 2 & \text{for } i = 1 \\ 1 & \text{for } 2 \leq i \leq 4. \end{cases}$$

By this definition we can find that the weight of all vertices are distinct and μ is defining a function $\mu: V \cup E \to \{1, 2, 3\}$. Such that, μ is defining a total vertex irregular $3-$labeling of Br_5. So that, $tvs(Br_5) \leq 3$. ■

3. The Total Edge Irregularity Strength of Diamond Graphs Br_5

In 2007, Bača et al. gave the lower bound of the total edge irregularity strength of any graph as follows.

***Theorem* 5.** [3] *Let $G = (V, E)$ be graph with maximum degree Δ, then*

$$tes(G) \geq max\left\{\left\lceil\frac{|E| + 2}{3}\right\rceil, \left\lceil\frac{\Delta + 1}{2}\right\rceil\right\},$$

where $|E|$ is the number of edges of G.

In Theorem 6, we determined the total edge irregularity strength of diamond graph on 10 vertices.

***Theorem* 6.** *The total edge irregularity strength of diamond graph Br_5 is* 8.

Proof. Since the number of edges of Br_5 is 20 and the maximum degree of Br_5 is 5, according to Theorem 4, we found that the total edge irregularity strength of diamond graph Br_5 bigger or equal to $\max\left\{\left\lceil\frac{22}{3}\right\rceil, \left\lceil\frac{6}{2}\right\rceil\right\} = 8$. To find the upper bound of the total edge irregularity strength of diamond graph Br_5 we have to construct a total edge irregular 8-labeling of diamond graph Br_5 as follows. Define a total labelling β on Br_5 as follows:

$$\beta(xv_i) = 8 \text{ for } 1 \leq i \leq 5,$$

$$\beta(v_iv_{i+1}) = \begin{cases} 2 & \text{for } i = 1,2 \\ 3 & \text{for } i = 3 \\ 6 & \text{for } i = 4, \end{cases}$$

$$\beta(v_iu_j) = \begin{cases} 1 & \text{for } (i,j) = (1,1),(2,1) \\ 2 & \text{for } (i,j) = (2,2),(3,3) \\ 3 & \text{for } (i,j) = (3,2),(4,3) \\ 5 & \text{for } (i,j) = (4,4),(5,4), \end{cases}$$

$$\beta(u_iu_{i+1}) = \begin{cases} 1 & \text{for } i = 1 \\ 4 & \text{for } i = 2,3, \end{cases}$$

$$\beta(x) = 8,$$

$$\beta(v_i) = i + 1 \text{ for } 1 \le i \le 5,$$

$$\beta(u_i) = \begin{cases} 1 & \text{for } i = 1,2 \\ 5 & \text{for } i = 3,4. \end{cases}$$

By this definition we can find that the weight of all edges are distinct and β is defining a function $\beta: V \cup E \to \{1, 2, \cdots, 8\}$. Such that, β is defining a total edge irregular $8-$labeling of Br_5. So that, $tes(Br_5) \le 8$. ■

4. The Total Irregularity Strength of Diamond Graphs Br_5

In 2013, Marzuki, Salman, and Miller gave the lower bound of the total irregularity strength of G as follows.

***Theorem* 7.** [7] *Let G be graph, then* $ts(G) \ge max\{tvs(G), tes(G)\}$.

***Theorem* 8.** *The total irregularity strength of diamond graph Br_5 is* 8.

Proof. Since the total vertex irregularity strength of diamond graphs and the total edge irregularity strength of diamond graphs Br_5 are 3 and 8, respectively, according to Theorem 7, we prove that the total irregularity strength of diamond graph Br_5 bigger or equal to 8. To find the total irregularity strength of diamond graph Br_5 smaller or equal to 8, we have to construct a totally irregular total 8-labeling of diamond graph Br_5 as follows. Define a total labelling θ on Br_5 as follows:

$$\theta(xv_i) = \begin{cases} 8 & \text{for } i = 1,2,3,5 \\ 7 & \text{for } i = 4, \end{cases}$$

$$\theta(v_i v_{i+1}) = \begin{cases} 2 & \text{for } i = 1,3 \\ 3 & \text{for } i = 2 \\ 5 & \text{for } i = 4, \end{cases}$$

$$\theta(v_i u_j) = \begin{cases} 1 & \text{for } (i,j) = (1,1),(2,1) \\ 2 & \text{for } (i,j) = (2,2),(3,3),(4,3) \\ 3 & \text{for } (i,j) = (3,2) \\ 5 & \text{for } (i,j) = (4,4) \\ 4 & \text{for } (i,j) = (5,4), \end{cases}$$

$$\theta(u_i u_{i+1}) = \begin{cases} 1 & \text{for } i = 1 \\ 3 & \text{for } i = 2 \\ 4 & \text{for } i = 3, \end{cases}$$

$$\theta(x) = 8,$$

$$\theta(v_i) = \begin{cases} i + 1 & \text{for } i = 1,2 \\ 4 & \text{for } i = 3 \\ 6 & \text{for } i = 4,5, \end{cases}$$

$$\theta(u_i) = \begin{cases} 1 & \text{for } i = 1,2 \\ 5 & \text{for } i = 3,4. \end{cases}$$

By this definition we can find that the weights of every edges are different and the weights of all vertices are distinct. Besides that, the total labelling θ is defining a function $\theta: V \cup E \rightarrow \{1, 2, \cdots, 8\}$. Such that, function θ is defining a totally irregular total $8-$labeling of Br_5. So that, $ts(Br_5) \leq 8$. ∎

There are many problems related to irregular labeling. Therefore, we provide the following problems.

Problem 9. What is the irregularity strength of diamond graph Br_n for $n \geq 6$?

Problem 10. What is the total edge irregularity strength of diamond graph Br_n for $n \geq 6$?

Problem 11. What is the total vertex irregularity strength of diamond graph Br_n for $n \geq 6$?

Problem 12. What is the total irregularity strength of diamond graph Br_n for $n \geq 6$?

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