JURNAL MATEMATIKA, STATISTIKA DAN KOMPUTASI e–ISSN: 2614–8811
Published by Department of Mathematics, Hasanuddin University, Indonesia
https://journal.unhas.ac.id/index.php/jmsk/index
Vol. 19, No. 1, September 2022, pp. 195 - 209
DOI: 10.20956/j.v19i1.21136

Shifted Liu-Type Estimator in The Linear Regression

Funda Erdugan*

* *Department of Statistics, University of Kırıkkale, Turkey,*
***Email address**: ferdugan@kku.edu.tr*

Abstract

The methods to solve the problem of multicollinearity have an important issue in the linear regression. The Liu-type estimator is one of these methods used to reduce its effect. This estimator is an estimator with two parameters denoted k and d. Kurnaz and Akay (2015) [6] introduced a new approach for the Liu-type estimator and called it new Liu-type (NL) estimator. This proposed estimator is based on a continuous function of k rather than two parameters and includes OLS, ridge estimator, Liu estimator, and some estimators with two biasing parameters as special cases. This study aimed to improve the NL estimator by shifting. The performance of the shifted NL estimator is compared to the NL estimator and other estimators depending on the mean squared error (MSE) criterion. The real data example and simulation study reveal that the SNL estimator can be a good selection in the linear regression model.

Keywords: Multicollinearity, Biased estimation, Ridge estimator, Liu estimator

1. INTRODUCTION

Consider the linear regression model

$$y = X\beta + \varepsilon \tag{1.1}$$

where y is an $n \times 1$ vector of observations on the dependent variable; X is a $n \times p$ matrix of the explanatory variable; β is an $p \times 1$ vector of unknown parameters; and ε is a $n \times 1$ vector of errors, with expectation $E(\varepsilon) = 0$ and $cov(\varepsilon) = \sigma^2 I$. The ordinary least squares (OLS) estimator of model (1.1) is given as $\hat{\beta} = (X'X)^{-1}X'y$.

Multicollinearity is one of the most significant problems in regression analysis. The OLS estimate is unstable and often gives rise to misleading information in the presence of multicollinearity, despite the fact that it is unbiased. Also, the variance of the estimator may be very large. Several biased estimators have been proposed as alternatives to the OLS estimator to tackle the multicollinearity problem [2, 14]. More stable estimations can be obtained by adding a biasing parameter. Hoerl and Kennard [3, 4] introduced the ridge estimator as an alternative to the OLS estimator. The ridge estimates are biased but have a smaller mean square error than OLS estimates [1]. It is given as

$$\hat{\beta}_k = (X'X + kI)^{-1}X'y,\ k > 0$$

where k is a biasing parameter. It is one of the most popular biased estimators and attempts to overcome the collinearity problem by adding a small biasing parameter k to the diagonal elements of the $X'X$ matrix. However, the ridge estimator is a nonlinear function of k. To resolve this problem, Liu [7] combined Stein's estimator [12] with the ridge estimator and proposed a new biased estimator of β, which is known as the Liu estimator

$$\hat{\beta}_d = F_d\hat{\beta},\quad F_d = (X'X + I)^{-1}(X'X + dI),\quad 0 < d < 1$$

where d is a biasing parameter. The advantage of the Liu estimator is that the d parameter is a linear function. Thus, the choice of d in the Liu estimator becomes easier than the selection of k in the ridge estimator. Liu [8] introduced the Liu-type estimator as

$$\hat{\beta}_{k,d} = (X'X + kI)^{-1}\left(X'y - d\hat{\beta}^*\right)$$

where $k > 0$, $-\infty < d < \infty$ and $\hat{\beta}^*$ can be any estimator of β. The Liu-type estimator has two biasing parameters: k and d. The first parameter k can be used exclusively to control the conditioning of $X'X + kI$, while the second parameter d is used to improve the fit and statistical properties. Özkale and Kaçıranlar [10] introduced the two-parameter (TP) estimator

$$\tilde{\beta}(k, d) = (X'X + kI)^{-1}\left(X'y + kd\hat{\beta}\right),\ k > 0,\ 0 < d < 1$$

which has general applications and includes the OLS, ridge and Liu estimators as special cases. Sakallıoğlu and Kaçıranlar [11] proposed $\hat{\beta}(k, d)$ as a general estimator for β, expressed as

$$\hat{\beta}_{SK}(k, d) = (X'X + I)^{-1}\left(X'y + d\hat{\beta}_k\right)$$

where $k > 0$, $-\infty < d < \infty$. This is called the $k - d$ class estimator, which is obtained by augmenting the equation $d\hat{\beta}_k = \beta + \varepsilon'$ to equation (1.1). Yang and Chang [16] introduced a new two-parameter estimator for β, which is given by

$$\hat{\beta}_{YC}(k, d) = F_d\hat{\beta}_k$$

where $k > 0$, $0 < d < 1$. $\hat{\beta}_{YC}(k, d)$ is obtained by augmenting the equation $(d - k)\hat{\beta}_k = \beta + \varepsilon'$ to equation (1.1). Kurnaz and Akay [6] suggested an alternative approach based on a function of the biasing parameter k. The NL estimator is defined as

$$\hat{\beta}_{NL} = (X'X + kI)^{-1}\left(X'y + f(k)\hat{\beta}^*\right)$$

where $f(k)$ is a continuous function of the biasing parameter k and $\hat{\beta}^*$ is any estimator of β. This estimator was obtained by augmenting the equation $\frac{f(k)}{k^{1/2}}\hat{\beta}^* = k^{1/2}\beta + \varepsilon'$ to equation (1.1). The $\hat{\beta}_{NL}$ estimator is a general estimator for β, including the OLS estimator, ridge estimator, Liu estimator and some estimators with two biasing parameters.

Researchers aim to obtain an estimator that has an optimal performance. It is preferable that estimators have a minimum MSE. The appropriate selection of the biasing parameter(s) in biased estimators may affect the MSE. Moreover, the selection of k and d may adversely affect the performance of the regression coefficients when the estimators have two biasing parameters. It is considered that it can be advantageous to use an appropriate function which depends on one parameter, as in the NL estimator. Kurnaz and Akay (2015) [6] have shown that the NL estimator is a general estimator and have given superiority conditions. The focus of this study is to improve the NL estimator via shifting the $f(k)$ function according to the ridge biasing parameter k and to investigate the effect of multicollinearity on MSE.

This article is organized as follows. In Section 2, the proposed estimator is introduced and some of its properties are discussed. In Section 3, the MSE matrix of the proposed estimator is derived and its mean squared error properties are shown to be superior to those of the other estimators. In Section 4, the form of the function $f(k)$ is determined, while Section 5 and Section 6 considers a numerical example and simulation study.

2. PROPOSED ESTIMATOR

To obtain more performance, we propose shifting the NL (SNL) estimator. It is defined as

$$\hat{\beta}_{SNL} = (X'X + kI)^{-1}\left(X'y + (f(k) - k)\hat{\beta}^*\right), k > 0$$

where $f(k)$ is a continuous function of the biasing parameter k. The estimator is obtained by augmenting the equation $\frac{f(k)-k}{k^{1/2}}\hat{\beta}^* = k^{1/2}\beta + \varepsilon'$ to equation (1.1) and then using the LS method.

It is now possible to select $f(k)$ as a linear function of the biasing parameter, such as $f(k) = ak + b$ where $a, b \in \mathbb{R}$, and from the definition of $\hat{\beta}_{SNL}$ we can see that it is a general estimator, including the OLS estimator, ridge estimator, Liu estimator and some estimators with two biasing parameters as special cases:

1. $\hat{\beta}_{SNL} = \hat{\beta}$, for $f(0) = 0$, the OLS estimator
2. $\hat{\beta}_{SNL} = \hat{\beta}_k$, for $f(k) = k$, the Ridge estimator
3. $\hat{\beta}_{SNL} = \hat{\beta}_d$, for $f(1) = a + b$ and $\hat{\beta}^* = \hat{\beta}$, the Liu estimator, where $a + b - 1$ corresponds to the biasing parameter d;
4. $\hat{\beta}_{SNL} = \hat{\beta}_{k,d}$, for $f(k) = k + b$, the Liu-type estimator of [8], where b corresponds to the biasing parameter $-d$;
5. $\hat{\beta}_{SNL} = \tilde{\beta}(k, d)$, for $f(k) = ak$ and $\hat{\beta}^* = \hat{\beta}$, the TP estimator, where $a - 1$ corresponds to the biasing parameter d;

Also, when we take $\hat{\beta}^* = \hat{\beta}_k$, proposed estimator can be rewritten as

$$\hat{\beta}_{SNL} = (X'X + kI)^{-1}(X'X + f(k)I)\hat{\beta}_k.$$

This estimator includes $k - d$ class estimator and the two-parameter estimator proposed Yang and Chang [16] respectively for $f(k) = k + b$ and $f(k) = b$ where b corresponds to the biasing parameter d.

Model (1.1) can be rewritten in canonical form as

$$y = Z\alpha + \varepsilon \tag{2.1}$$

where $Z = XQ$, $\alpha = Q'\beta$, and Q is the orthogonal matrix whose columns constitute the eigenvectors of $X'X$. It can then be shown that $Z'Z = Q'X'XQ = \Lambda = diag(\lambda_1, \ldots, \lambda_p)$, where $\lambda_1 \geq \ldots \geq \lambda_p > 0$ are the ordered eigenvalues of $X'X$. For model (2.1), we can obtain the representations

$$\hat{\alpha} = \Lambda^{-1}Z'y,$$

$$\hat{\alpha}_{NL} = (\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}Z'y, \text{ for } \hat{\alpha}^* = \hat{\alpha},$$

$$\hat{\alpha}^*_{NL} = (\Lambda + kI)^{-1}\left(\Lambda + \left(k + f(k)\right)I\right)(\Lambda + kI)^{-1}Z'y, \text{ for } \hat{\alpha}^* = \hat{\alpha}_k.$$

The canonical forms of the proposed estimator are given as respectively for $\hat{\alpha}^* = \hat{\alpha}$ and for $\hat{\alpha}^* = \hat{\alpha}_k$

$$\hat{\alpha}_{SNL} = (\Lambda + kI)^{-1}(\Lambda + (f(k) - k)I)\Lambda^{-1}Z'y$$

and

$$\hat{\alpha}^*_{SNL} = (\Lambda + kI)^{-1}(\Lambda + f(k)I)(\Lambda + kI)^{-1}Z'y.$$

3. PROPERTIES OF SNL ESTIMATOR

The MSE criterion is widely used as a measure of the closeness of the estimates to the parameter β. This criterion enables to decide the performance of the estimator. The matrix MSE of $\tilde{\beta}$ for β is $MSE(\tilde{\beta}) = E(\tilde{\beta} - \beta)'(\tilde{\beta} - \beta) = cov(\tilde{\beta}) + bias(\tilde{\beta})'bias(\tilde{\beta})$ where $bias(\tilde{\beta}) = E(\tilde{\beta}) - \beta$ is the bias vector of the $\tilde{\beta}$ estimator and $cov(\tilde{\beta})$ is the variance-covariance matrix of the $\tilde{\beta}$ estimator. The scalar mean square error (*mse*) is $mse(\tilde{\beta}) = tr\left(MSE(\tilde{\beta})\right)$, where tr denotes trace. In the case of two estimators $\tilde{\beta}_1$ and $\tilde{\beta}_2$ of β, the superiority of $\tilde{\beta}_1$ with respect to $\tilde{\beta}_2$ depends on whether the difference between the estimators' MSE matrix is positive definite (pd). The following lemma is of significance for the comparisons given.

Lemma 3.1. (Trenkler and Toutenburg [13]) *Let $\hat{\beta}_1 = A_1 y$, $\hat{\beta}_2 = A_2 y$ be two linear estimators of β such that $D = (A_1A_1' - A_2A_2') > 0$. Then $MSE(\hat{\beta}_1) - MSE(\hat{\beta}_2) = \sigma^2 D + b_1b_1' - b_2b_2' > 0$ if and only if $b_2'(\sigma^2 D + b_1b_1')^{-1}b_2 < 1$ where $MSE(\hat{\beta}_i) = cov(\hat{\beta}_i) + b_ib_i'$, $b_i = bias(\hat{\beta}_i) = (A_iX - I)\beta$, $i = 1,2$.*

The biasing vector and covariance matrix of the estimator $\hat{\alpha}_{SNL}$ for $\hat{\alpha}^* = \hat{\alpha}$ are given as

$$bias(\hat{\alpha}_{SNL}) = (f(k) - 2k)(\Lambda + kI)^{-1}\alpha,$$

$$cov(\hat{\alpha}_{SNL}) = \sigma^2(\Lambda + kI)^{-1}(\Lambda + (f(k) - k)I)\Lambda^{-1}(\Lambda + (f(k) - k)I)(\Lambda + kI)^{-1}.$$

Thus, the MSE matrix of the estimator $\hat{\alpha}_{SNL}$ is obtained as

$$MSE(\hat{\alpha}_{SNL}) = \sigma^2(\Lambda + kI)^{-1}(\Lambda + (f(k) - k)I)\Lambda^{-1}(\Lambda + (f(k) - k)I)(\Lambda + kI)^{-1} + (f(k) - 2k)^2(\Lambda + kI)^{-1}\alpha\alpha'(\Lambda + kI)^{-1} \quad (3.1)$$

Similarly, the MSE matrix of the estimator $\hat{\alpha}^*_{SNL}$ is given by

$$MSE(\hat{\alpha}^*_{SNL}) = \sigma^2(\Lambda + kI)^{-1}(\Lambda + f(k)I)(\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1} \times (\Lambda + f(k)I)(\Lambda + kI)^{-1} + bias(\hat{\alpha}^*_{SNL})bias(\hat{\alpha}^*_{SNL})' \quad (3.2)$$

where $bias(\hat{\alpha}^*_{SNL}) = [(\Lambda + kI)^{-1}(\Lambda + f(k)I)(\Lambda + kI)^{-1}\Lambda - I]\alpha$ for $\hat{\alpha}^* = \hat{\alpha}_k$.

Comparison of SNL and OLS

It is known that the MSE matrix of the OLS estimator is given as

$$MSE(\hat{\alpha}) = \sigma^2\Lambda^{-1}. \quad (3.3)$$

The difference between the MSE matrices is used to compare the OLS estimator and the $\hat{\alpha}_{SNL}$ estimator, using equation (3.1) and equation (3.3).

Theorem 3.1 *Let $k > 0$ and $-2\lambda_i < f(k) < 2k$ for all $i = 1, \dots, p$. Then $MSE(\hat{\alpha}) - MSE(\hat{\alpha}_{SNL})$ is pd if and only if*

$$(f(k) - 2k)^2\alpha'(\Lambda + kI)^{-1}{D_1}^{-1}(\Lambda + kI)^{-1}\alpha < \sigma^2.$$

Proof. The difference between the covariance matrices for the OLS estimator and new estimator is

$$D_1 = [\Lambda^{-1} - (\Lambda + kI)^{-1}(\Lambda + (f(k) - k)I)\Lambda^{-1}(\Lambda + (f(k) - k)I)(\Lambda + kI)^{-1}]. \quad (3.4)$$

The trace operator can be applied to this difference matrix (3.3). This gives

$$tr(D_1) = \left\{\frac{(\lambda_i + k)^2 - \left(\lambda_i + (f(k) - k)\right)^2}{\lambda_i(\lambda_i + k)^2}\right\}_{i=1}^{p}.$$

The sign of equation (3.4) directly depends on whether the expression $(\lambda_i + k)^2 - \left(\lambda_i + (f(k) - k)\right)^2$ is positive or negative. A necessary and sufficient condition for this expression to be positive is $-2\lambda_i < f(k) < 2k$ for $k > 0$ and for all $i = 1, \ldots, p$. This theorem is proved by application of the Lemma 3.1.

Comparison of SNL and NL

Firstly, we state the MSE matrices of NL. The MSE matrices are given for $\hat{\alpha}^* = \hat{\alpha}$ and $\hat{\alpha}^* = \hat{\alpha}_k$ respectively

$$MSE(\hat{\alpha}_{NL}) = \sigma^2(\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + f(k)I)(\Lambda + kI)^{-1} + (f(k) - k)^2(\Lambda + kI)^{-1}\,\alpha\,\alpha'(\Lambda + kI)^{-1} \quad (3.5)$$

and

$$MSE(\hat{\alpha}^*_{NL}) = \sigma^2(\Lambda + kI)^{-1}\left(\Lambda + \left(k + f(k)\right)I\right)(\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1}\left(\Lambda + \left(k + f(k)\right)I\right)(\Lambda + kI)^{-1} + bias(\hat{\alpha}^*_{NL})bias(\hat{\alpha}^*_{NL})' \quad (3.6)$$

where $bias(\hat{\alpha}^*_{NL}) = (\Lambda + kI)^{-1}\left(\Lambda + \left(k + f(k)\right)I\right)(\Lambda + kI)^{-1}\Lambda\,\alpha$.
Then, we can express the following theorems.

Theorem 3.2 *If $k > 0$ and $f(k) > \frac{k}{2} - \lambda_i$ for all $i = 1, \ldots, p$, then $\hat{\alpha}_{SNL}$ estimator is superior to the $\hat{\alpha}_{NL}$ estimator in the sense of the MSE criterion if and only if*

$$(f(k) - 2k)^2\alpha'(\Lambda + kI)^{-1}[D_2 + (f(k) - k)^2(\Lambda + kI)^{-1}\,\alpha\,\alpha'(\Lambda + kI)^{-1}](\Lambda + kI)^{-1}\alpha < 1.$$

Proof. Using the covariance matrices of the estimators $\hat{\alpha}_{NL}$ and $\hat{\alpha}_{SNL}$, the difference matrix is

$$D_2 = \sigma^2[(\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + f(k)I)(\Lambda + kI)^{-1} - (\Lambda + kI)^{-1}(\Lambda + (f(k) - k)I)\Lambda^{-1}(\Lambda + (f(k) - k)I)(\Lambda + kI)^{-1}] \quad (3.7)$$

From equation (3.7), it can be seen

$$tr(D_2) = \sigma^2\left\{\frac{\left(\lambda_i + f(k)\right)^2 - (\lambda_i + f(k) - k)^2}{\lambda_i(\lambda_i + k)^2}\right\}_{i=1}^{p}$$

All λ_i are positive and equation (3.7) is a pd matrix when $\left(\lambda_i + f(k)\right)^2 - (\lambda_i + f(k) - k)^2 > 0$. This inequality is positive when $f(k) > \frac{k}{2} - \lambda_i$. Then by applying the Lemma 3.1, we prove the theorem.

Theorem 3.3 *If $k > 0$ and $f(k) > -\frac{k}{2} - \lambda_i$ for all $i = 1, \ldots, p$, then $\hat{\alpha}^*_{SNL}$ estimator is superior to the $\hat{\alpha}^*_{NL}$ estimator in the sense of the MSE criterion if and only if*

$$\alpha'\Lambda(\Lambda + kI)^{-1}\left(\Lambda + \left(k + f(k)\right)I\right)(\Lambda + kI)^{-1}$$
$$\times\left(D_3 + [(\Lambda + kI)^{-1}(\Lambda + f(k)I)(\Lambda + kI)^{-1}\Lambda - I]\alpha\alpha'^{[(\Lambda + kI)^{-1}(\Lambda + f(k)I)(\Lambda + kI)^{-1}\Lambda - I]}\right)$$
$$\times(\Lambda + kI)^{-1}\left(\Lambda + \left(k + f(k)\right)I\right)(\Lambda + kI)^{-1}\Lambda\,\alpha < 1$$

Proof. Taking account of the estimators $\hat{\alpha}^*_{NL}$ and $\hat{\alpha}^*_{SNL}$, we get the difference of covariance matrices as

$$D_3 = \sigma^2\big[(\Lambda + kI)^{-1}\big(\Lambda + \big(k + f(k)\big)I\big)(\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1}\big(\Lambda + \big(k + f(k)\big)I\big)(\Lambda + kI)^{-1} - (\Lambda + kI)^{-1}(\Lambda + f(k)I)(\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1}(\Lambda + f(k)I)(\Lambda + kI)^{-1}\big].$$

If the trace of D_3 is taken, equation (3.8) is obtained

$$tr(D_3) = \sigma^2\lambda_i\left\{\frac{(\lambda_i + f(k) + k)^2 - \big(\lambda_i + f(k)\big)^2}{(\lambda_i + k)^4}\right\}_{i=1}^{p}. \tag{3.8}$$

It can be seen that the $\hat{\alpha}_{SNL}^*$ estimator is superior to the $\hat{\alpha}_{NL}^*$ estimator when $(\lambda_i + f(k) + k)^2 - \big(\lambda_i + f(k)\big)^2 > 0$. This inequality is positive for $k > 0$ and $f(k) > -\frac{k}{2} - \lambda_i$. By applying the Lemma 3.1, this theorem is completed.

4. SELECTION OF THE $f(k)$ FUNCTION

Researchers have developed several approaches for estimating the optimal values of the biasing parameters d or k in literature. Since the proposed estimator depends on the function of the biasing parameter k, selection of this function is an important factor. The problem of determining the $f(k)$ function are deal with in this section. The diagonal elements of the MSE matrix given by equation (3.1) are

$$mse(\hat{\alpha}_{SNL}) = \sum_{i=1}^{p}\left(\sigma^2\frac{(\lambda_i+f(k)-k)^2}{\lambda_i(\lambda_i+k)^2} + \alpha_i^2\frac{(f(k)-2k)^2}{(\lambda_i+k)^2}\right). \tag{4.1}$$

Equation (4.1) can be considered as a $g(k)$ function. By differentiating $g(k)$ function with respect to k, we obtain

$$g'(k) = \sum_{i=1}^{p}\frac{2\sigma^2}{\lambda_i(\lambda_i + k)^3}(\lambda_i + f(k) - k)[(f'(k) - 1)(\lambda_i + k) - (\lambda_i + f(k) - k)] + \frac{2\alpha_i^2(f(k) - 2k)}{(\lambda_i + k)^3}[(f'(k) - 2)(\lambda_i + k) - (f(k) - 2k)]$$

or

$$g'(k) = \sum_{i=1}^{p}\frac{2\sigma^2}{\lambda_i(\lambda_i + k)^3}(\lambda_i + f(k) - k)[(f'(k) - 1)(\lambda_i + k) - (\lambda_i + f(k) - k)] + \frac{2\alpha_i^2}{(\lambda_i + k)^3}(f(k) - 2k)[(f'(k) - 1)(\lambda_i + k) - (\lambda_i + f(k) - k)].$$

After some algebraic simplifications, the derivative of $mse(\hat{\alpha}_{SNL})$ with respect to k is

$$g'(k) = \sum_{i=1}^{p}\frac{2}{\lambda_i(\lambda_i + k)^3}[(f'(k) - 1)(\lambda_i + k) - (\lambda_i + f(k) - k)] \times \big[\sigma^2(\lambda_i + f(k) - k) + \alpha_i^2\lambda_i(f(k) - 2k)\big].$$

Let $g'(k) = 0$. Setting the derivative to zero gives rise to the following.

Fact 1. $(f'(k) - 1)(\lambda_i + k) - (\lambda_i + f(k) - k) = 0$ is obtained. Integrating this equation gives

$$f(k) = c_1k + (c_1 - 2)\lambda_i,\ i = 1, \ldots, p,$$

where c_1 is the integration constant.

Fact 2. $\sigma^2(\lambda_i + f(k) - k) + \alpha_i^2\lambda_i(f(k) - 2k) = 0$ is obtained. Solving this equation, it can be seen that

$$f(k)=\frac{(\sigma^2+2\alpha_i^2\lambda_i)}{\sigma^2+\alpha_i^2\lambda_i}k-\frac{\sigma^2\lambda_i}{\sigma^2+\alpha_i^2\lambda_i},\ i=1,\ldots,p.$$

The function $f(k)$ is a linear function of the biasing parameter k. If we take the form $f(k)=ak+b$, where $a,b\in\mathbb{R}$, the optimal k can be obtained as

$$k=\frac{b(\sigma^2+\alpha_i^2\lambda_i)+\sigma^2\lambda_i}{(\sigma^2+2\alpha_i^2\lambda_i)-a(\sigma^2+\alpha_i^2\lambda_i)},\qquad (4.2)$$

where $a\neq\frac{(\sigma^2+2\alpha_i^2\lambda_i)}{\sigma^2+\alpha_i^2\lambda}$ and $i=1,\ldots,p$. Since σ^2 and α_i in equation (4.2) are unknown, they are replaced with their estimators respectively the residual mean square estimate and OLS. Thus, the estimate of the biasing parameter k is obtained as

$$k_{opt}=\frac{b(\hat{\sigma}^2+\hat{\alpha}_i^2\lambda_i)+\hat{\sigma}^2\lambda_i}{(\hat{\sigma}^2+2\hat{\alpha}_i^2\lambda_i)-a(\hat{\sigma}^2+\hat{\alpha}_i^2\lambda_i)}.\qquad (4.3)$$

Furthermore, when $a=1$ and $b=0$, k_{opt} in equation (4.3) is

$$k_{HK}=\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2},$$

which is the estimated value of k (HK) proposed by [3]. Similarly, we can achieve the estimate of d proposed by [7] for $k>0$ and $f(1)=a+1$ where a corresponds to the biasing parameter d. By taking derive of equation (4.1), we get

$$\hat{d}_{opt}=\frac{\sum_{i=1}^{p}\left(\hat{\alpha}_i^2-\hat{\sigma}^2\right)/(\lambda_i+1)^2}{\sum_{i=1}^{p}\left(\hat{\sigma}^2+\hat{\alpha}_i^2\lambda\right)/\lambda_i(\lambda_i+1)^2}.$$

To obtain form of the $f(k)$ function for $\hat{\alpha}_{SNL}^*$ estimator, we can write

$$g^*(k)=mse(\hat{\alpha}_{SNL}^*)=\sum_{i=1}^{p}\left(\sigma^2\frac{\lambda_i\left(\lambda_i+f(k)\right)^2}{(\lambda_i+k)^4}+\alpha_i^2\left(\frac{\lambda_i\left(\lambda_i+f(k)\right)}{(\lambda_i+k)^2}-1\right)^2\right).$$

The derivative of $g^*(k)$ with respect to k

$$g^{*\prime}(k)=\sum_{i=1}^{p}\frac{2\lambda_i}{(\lambda_i+k)^5}\left[f'(k)(\lambda_i+k)-2\left(\lambda_i+f(k)\right)\right]$$
$$\times\left[\sigma^2\left(\lambda_i+f(k)\right)+\alpha_i^2\left(\lambda_i\left(\lambda_i+f(k)\right)-(\lambda_i+k)^2\right)\right].$$

Then setting the results to zero we have Fact 3 and Fact 4.

Fact 3. $f'(k)(\lambda_i+k)-2\left(\lambda_i+f(k)\right)=0$ is obtained. Integrating this equation gives

$$f(k)=(\lambda_i+k)^2c_2-\lambda_i,\ i=1,\ldots,p,$$

where c_2 is the integration constant.

Fact 4: $\sigma^2\lambda_i\left(\lambda_i+f(k)\right)+\lambda_i\alpha_i^2\left(\lambda_i\left(\lambda_i+f(k)\right)-(\lambda_i+k)^2\right)=0$ is obtained. Solving this equation gives, it can be seen that

$$f(k)=\frac{\alpha_i^2k^2+2\lambda_i\alpha_i^2k-\sigma^2\lambda_i}{(\sigma^2+\alpha_i^2\lambda_i)},\ i=1,\ldots,p.$$

The function $f(k)$ is a quadratic function of the biasing parameter k. For $f(k)=a_1k^2+b_1k+z$, where $a_1,b_1,z\in\mathbb{R}$, the optimal k can be obtained as

$$k_{opt}=\frac{2\lambda_i\alpha_i^2-b_1(\sigma^2+\alpha_i^2\lambda_i)\mp\sqrt{\Delta}}{2\left(a_1(\sigma^2+\alpha_i^2\lambda_i)-\alpha_i^2\right)},\ a_1\neq\frac{\alpha_i^2}{(\sigma^2+\alpha_i^2\lambda_i)}$$

where $\Delta=\left(b_1\left(\sigma^2+\alpha_i^2\lambda_i\right)-2\lambda_i\alpha_i^2\right)^2-4\left(a_1\left(\sigma^2+\alpha_i^2\lambda_i\right)-\alpha_i^2\right)\left(z\left(\sigma^2+\alpha_i^2\lambda_i\right)+\sigma^2\lambda_i\right)$. For $\Delta\geq0$ and the estimators of σ^2 and α_i, we can obtain

$$\hat{k}_{opt} = \frac{2\lambda_i\hat{\alpha}_i^2 - b_1(\hat{\sigma}^2 + \hat{\alpha}_i^2\lambda_i) \mp \sqrt{\Delta}}{2(a_1(\hat{\sigma}^2 + \hat{\alpha}_i^2\lambda_i) - \hat{\alpha}_i^2)}.$$

5. REAL DATA EXAMPLE

In this section, a real data example is considered to illustrate the performance of the SNL estimator. The Portland cement (Woods et al., [15]) data set involves of four explanatory variables and each variable consists of 13 observations. We considered the linear model with intercept for this data. The model is affected by severe collinearity ($\kappa(X) = 6056.37$), so that X may be considered as ill-conditioned. It means that the OLS estimator should not be used.

To compare the performance of the SNL estimator, it is necessary to calculate the scalar mean square error values of all the variables examined. This is accomplished by replacing the corresponding unknown model parameters with their least squares estimates. The following estimators are compared in this example:

$\hat{\beta}$: OLS estimator
$\hat{\beta}_k$: Ridge estimator
$\hat{\beta}_d$: Liu estimator
$\hat{\beta}_{k,d}$: Liu-type estimator of Liu (2003) for $\hat{\beta}^* = \hat{\beta}$
$\hat{\beta}^*_{k,d}$: Liu-type estimator of Liu (2003) for $\hat{\beta}^* = \hat{\beta}_k$
$\tilde{\beta}(k,d)$: TP estimator
$\hat{\beta}_{SK}(k,d)$: $k-d$ class estimator
$\hat{\beta}_{YC}(k,d)$: Two-parameter estimator of Yang and Chang [16]
$\hat{\beta}_{NL}$: Liu-type estimator of Kurnaz and Akay [6] for $\hat{\beta}^* = \hat{\beta}$
$\hat{\beta}^*_{NL}$: Liu-type estimator of Kurnaz and Akay [6] for $\hat{\beta}^* = \hat{\beta}_k$
$\hat{\beta}_{SNL}$: Shifted NL estimator for $\hat{\beta}^* = \hat{\beta}$
$\hat{\beta}^*_{SNL}$: Proposed estimator for $\hat{\beta}^* = \hat{\beta}_k$

The estimated parameters and the *mse* values of the estimators are presented in Tables 1-8.

Table 1. Estimated values of parameters and MSE for $k_{HK} = 0.0015$ and $d = 0.95$

	$f(k)$	β_0	β_1	β_2	β_3	β_4	*mse*
$\hat{\beta}$	-	62.4054	1.5511	0.5102	0.1019	-0.1441	4912.0902
$\hat{\beta}_k$	-	27.6342	1.9088	0.8685	0.4677	0.2072	2170.9613
$\hat{\beta}_d$	-	59.2913	1.5825	0.5424	0.1342	-0.1125	4443.4372
$\hat{\beta}_{k,d}$	-	-21488.8573	223.2307	222.6230	226.8463	217.5551	1.048x10^9
$\tilde{\beta}(k,d)$	-	60.6668	1.5690	0.5281	0.1202	-0.1265	4645.0354
$\hat{\beta}_{NL}$	I	29.3442	1.8912	0.8509	0.4498	0.1899	2177.7924
$\hat{\beta}_{NL}$	III	62.3868	1.5513	0.5104	0.1021	-0.1439	4909.1726
$\hat{\beta}_{SNL}$	III	27.6157	1.9090	0.8687	0.4679	0.2074	**2170.9597**

Table 2. Estimated values of parameters and MSE for $d = 0.95$

	$f(k)$	k	β_0	β_1	β_2	β_3	β_4	*mse*
$\hat{\beta}$	-	-	62.4054	1.5511	0.5102	0.1019	-0.1441	4912.0902
$\hat{\beta}_k$	-	0.00065	40.7072	1.7743	0.7338	0.3302	0.0751	2559.4448
$\hat{\beta}_d$	-	-	59.2913	1.5825	0.5424	0.1342	-0.1125	4443.4372
$\hat{\beta}_{k,d}$	-	0.00065	-31672.0592	327.9829	327.5722	333.9900	320.4193	2.27x10^9
$\tilde{\beta}(k,d)$	-	0.00065	61.3205	1.5623	0.5213	0.1133	-0.1331	4743.8350
$\hat{\beta}_{NL}$	I	0.0407	27.6068	1.9087	0.8689	0.4678	0.2075	**2170.9595**
$\hat{\beta}_{NL}$	III	0.00065	49.2911	1.6860	0.6453	0.2399	-0.0116	3235.3495
$\hat{\beta}_{SNL}$	III	0.00065	27.6068	1.9091	0.8688	0.4680	0.2075	**2170.9595**

Table 3. Estimated values of parameters and MSE for $k_{HK} = 0.0015$ and $d = 0.95$

	$f(k)$	β_0	β_1	β_2	β_3	β_4	*mse*
$\hat{\beta}^*_{k,d}$	-	-9491.1602	99.8132	98.9734	100.6113	96.3620	1.7471x10^{11}
$\hat{\beta}_{SK}(k,d)$	-	26.2988	1.9218	0.8824	0.4813	0.2208	2174.8318
$\hat{\beta}_{YC}(k,d)$	-	26.2565	1.9222	0.8829	0.4817	0.2212	2175.0863
$\hat{\beta}^*_{NL}$	IV	42.8083	1.7527	0.7121	0.3081	0.0539	2694.0545
$\hat{\beta}^*_{NL}$	II	28.4791	1.9001	0.8598	0.4589	0.1986	2172.6823
$\hat{\beta}^*_{SNL}$	IV	27.4257	1.9109	0.8707	0.4699	0.2093	**2171.0338**

Table 4. Estimated values of parameters and MSE for $d = 0.95$

	$f(k)$	k	β_0	β_1	β_2	β_3	β_4	*mse*
$\hat{\beta}^*_{k,d}$	-	0.0064	-1233.3139	14.8666	13.8671	13.7254	12.9465	1.7451x10^{11}
$\hat{\beta}_{SK}(k,d$	-	0.0064	9.5834	2.0938	1.0547	0.6572	0.3896	2906.2846
$\hat{\beta}_{YC}(k,d$	-	0.0064	9.5196	2.0944	1.0554	0.6578	0.3903	2911.4941
$\hat{\beta}^*_{NL}$	IV	0.0064	35.9832	1.8231	0.7824	0.3800	0.1228	2329.7854
$\hat{\beta}^*_{NL}$	II	0.0139	27.6071	1.9098	0.8686	0.4686	0.2073	**2170.9596**
$\hat{\beta}^*_{SNL}$	IV	0.0064	27.6070	1.9092	0.8688	0.4681	0.2074	**2170.9596**

Table 5. Estimated values of parameters and MSE for $k_{HK} = 0.0015$ and $d = 0.1$

	$\boldsymbol{f(k)}$	β_0	β_1	β_2	β_3	β_4	*mse*

$\hat{\beta}$	-	62.4054	1.5511	0.5102	0.1019	-0.1441	4912.0902
$\hat{\beta}_k$	-	27.6342	1.9088	0.8685	0.4677	0.2072	2170.9613
$\hat{\beta}_d$	-	6.3512	2.1154	1.0907	0.6828	0.4240	3193.6486
$\hat{\beta}_{k,d}$	-	-2237.2596	25.2058	24.2111	24.2971	23.0859	$1.1614\text{x}10^7$
$\tilde{\beta}(k,d)$	-	31.1113	1.8730	0.8327	0.4312	0.1721	2198.7615
$\hat{\beta}_{NL}$	I	29.3442	1.8912	0.8509	0.4498	0.1899	2177.7924
$\hat{\beta}_{NL}$	III	62.3868	1.5513	0.5104	0.1021	-0.1439	4909.1726
$\hat{\beta}_{SNL}$	III	27.6157	1.9090	0.8687	0.4679	0.2074	**2170.9597**

Table 6. Estimated values of parameters and MSE for $d = 0.1$

	$f(k)$	k	β_0	β_1	β_2	β_3	β_4	*mse*
$\hat{\beta}$	-		62.4054	1.5511	0.5102	0.1019	-0.1441	4912.0902
$\hat{\beta}_k$	-		40.7072	1.7743	0.7338	0.3302	0.0751	2559.4448
$\hat{\beta}_d$	-		6.3512	2.1154	1.0907	0.6828	0.4240	3193.6486
$\hat{\beta}_{k,d}$	-		- 3297.4788	36.1120	35.1378	35.4523	33.7956	$2.5029\text{x}10^7$
$\tilde{\beta}(k,d)$	-		42.8770	1.7520	0.7114	0.3074	0.0532	2698.7921
$\hat{\beta}_{NL}$	I	0.0407	27.6068	1.9087	0.8689	0.4678	0.2075	**2170.9595**
$\hat{\beta}_{NL}$	III	0.00065	49.3050	1.6859	0.6452	0.2397	-0.0117	3236.7074
$\hat{\beta}_{SNL}$	III	0.00065	27.6068	1.9091	0.8688	0.4680	0.2075	**2170.9595**

Table 7. Estimated values of parameters and MSE for $k_{HK} = 0.0015$ and $d = 0.1$

	$f(k)$	β_0	β_1	β_2	β_3	β_4	*mse*
$\hat{\beta}^*_{k,d}$	-	-974.3441	12.2145	11.1954	11.0092	10.3287	1.7472×10^{11}
$\hat{\beta}_{SK}(k,d)$	-	2.8783	2.1511	1.1265	0.7193	0.4591	3555.1416
$\hat{\beta}_{YC}(k,d)$	-	2.8360	2.1515	1.1270	0.7198	0.4595	3559.8813
$\hat{\beta}^*_{NL}$	II	28.4791	1.9001	0.8598	0.4589	0.1986	2172.6823
$\hat{\beta}^*_{NL}$	IV	42.8083	1.7527	0.7121	0.3081	0.0539	2694.0545
$\hat{\beta}^*_{SNL}$	IV	27.4257	1.9109	0.8707	0.4699	0.2093	**2171.0338**

Table 8. Estimated values of parameters and MSE for $d = 0.1$

	$f(k)$	k	β_0	β_1	β_2	β_3	β_4	*mse*
$\hat{\beta}^*_{k,d}$	-		-	3.4348	2.3993	2.0291	1.7074	$1.7451\text{x}10^{11}$

			120.8594					
$\hat{\beta}_{SK}(k,d)$	-		1.1188	2.1692	1.1446	0.7379	0.4769	3759.1310
$\hat{\beta}_{YC}(k,d)$	-		1.0551	2.1698	1.1453	0.7385	0.4775	3766.7827
$\hat{\beta}^*_{NL}$	II	0.0139	27.6071	1.9098	0.8686	0.4686	0.2073	**2170.9596**
$\hat{\beta}^*_{NL}$	IV	0.0064	35.9832	1.8231	0.7824	0.3800	0.1228	2329.7854
$\hat{\beta}^*_{SNL}$	IV	0.0064	27.6070	1.9092	0.8688	0.4681	0.2074	**2170.9596**

The $f(k)$ function for the NL and SNL estimators is also obtained. The functions are numbered for convenience as follows:

Estimator	Function Number	Function	k_{opt}
$\hat{\beta}_{NL}$	I	$f(k) = 0.44k - 0.0006$	0.0407
$\hat{\beta}^*_{NL}$	II	$f(k) = 362k^2 - 0.11k - 0.0006$	0.0139
$\hat{\beta}_{SNL}$	III	$f(k) = 1.4424k - 0.00068$	0.00065
$\hat{\beta}^*_{SNL}$	IV	$f(k) = 363.2079k^2 + 0.8848k - 0.0007$	0.0064

Kurnaz and Akay [6] obtained $f(k) = 0.44k - 0.0006$ and $f(k) = 362k^2 - 0.11k - 0.0006$ respectively, for $\hat{\beta}_{NL}$ and $\hat{\beta}^*_{NL}$. Also, optimal k values are $k_{opt} = 0.0407$ and $k_{opt} = 0.0139$ respectively for $\hat{\beta}_{NL}$ and $\hat{\beta}^*_{NL}$. We get the $f(k) = 1.4424k - 0.00068$ and $f(k) = 363.2079k^2 + 0.8848k - 0.0007$ respectively, for $\hat{\beta}_{SNL}$ and $\hat{\beta}^*_{SNL}$. In this study, optimal k values are obtained $k_{opt} = 0.00065$ and $k_{opt} = 0.0064$ respectively for $\hat{\beta}_{SNL}$ and $\hat{\beta}^*_{SNL}$. Hoerl et al. [5] proposed a different estimator of k (HKB). That is $\hat{k}_{HKB} = \frac{p\hat{\sigma}^2}{\hat{\alpha}'\hat{\alpha}}$ where p is the number of regressors. Hoerl and Kennard (1970) [3] estimation of k and Hoerl et al. [5] estimation of k are considered in the comparison of the performances of the estimators in this study. Estimation of d is not a main consideration for this example; however, values of 0.1 and 0.95 were chosen for this parameter, in order to determine its effect on the performance of the new estimator.

From Tables 1-3-5 and 7 we can see that the *mse* values of the SNL estimator perform better than the other estimators. These superiorities are provided when k is selected as $k_{HK} = 0.0015$. It is also valid when the function $f(k)$ is linear or quadratic function. NL and SNL estimators displayed the same performance when they used their k optimal values.

Table 9. Estimated values of parameters and MSE for $k_{HKB} = 0.0077$ and $d = 0.95$

	$f(k)$	β_0	β_1	β_2	β_3	β_4	*mse*
$\hat{\beta}_{NL}$	I	28.0589	1.9043	0.8642	0.4632	0.2029	2171.4222
$\hat{\beta}_{NL}$	III	81.4699	1.3550	0.3137	-0.0987	-0.3366	8738.3004
$\hat{\beta}_{SNL}$	III	27.6287	1.9088	0.8686	0.4678	0.2072	**2170.9606**
$\hat{\beta}^*_{NL}$	II	27.6798	1.9085	0.8680	0.4674	0.2067	2170.9716
$\hat{\beta}^*_{NL}$	IV	34.9680	1.8336	0.7929	0.3908	0.1331	2293.6215
$\hat{\beta}^*_{SNL}$	IV	27.6144	1.9092	0.86869	0.46809	0.20735	**2170.9597**

From Table 9, if the $f(k)$ function is selected in Kurnaz and Akay [6] ($f(k) = I$), mse value of NL estimator is obtained 2171.4222. For $f(k) = III$, we get mse value of SNL estimator is 2170.9606. Thus, $mse(\hat{\beta}_{SNL}) = 2170.9606 < mse(\hat{\beta}_{NL}) = 2171.4222$. We can say that the results for $\hat{\beta}^*_{NL}$ and $\hat{\beta}^*_{NL}$ are similar to obtained from $\hat{\beta}_{NL}$ and $\hat{\beta}_{SNL}$. For k_{HKB}, $mse(\hat{\beta}^*_{SNL}) =$

$2170.9597 < mse(\hat{\beta}_{NL}^{*}) = 2170.9716$, when $f(k)$ is selected as IV and II respectively. Moreover, in Figure 1 is shown an estimated MSE values of the estimators.

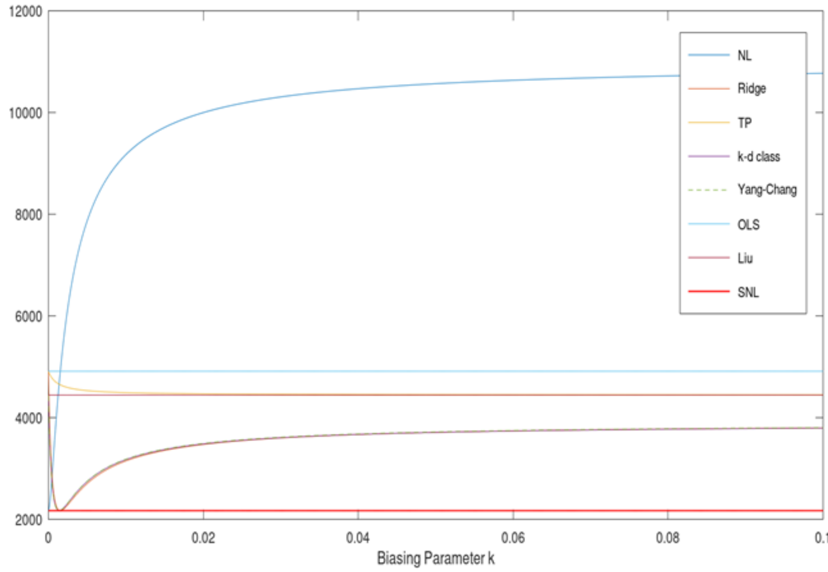

Figure 1. The scalar mean square error values of the estimators

6. SIMULATION RESULTS

In this section, to compare the proposed estimators we carried out a simulation study. Our simulation model is based on [9]. The explanatory variables are generated by

$$x_{ij} = (1-\rho^2)^{\frac{1}{2}} z_{ij} + \rho z_{i(p+1)}, \; i = 1, \ldots, n; j = 1, \ldots, p$$

where z_{ij} are independent standard normal pseudo-random numbers and ρ is specified so that the correlation between any two explanatory variables is given by ρ^2. $p = 5$ is the number of the explanatory variables and we have simulated the data with sample sizes $n = 30$ and 100. By choosing the true coefficient vector β as the normalized eigenvector corresponding to the largest eigenvalue of the $X'X$ matrix, the dependent variable y_i, $i = 1,2, \ldots, n$ are then generated by the following equation

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_5 x_{i5} + \varepsilon_i, \quad i = 1, \ldots, n$$

where ε_i are independent normal pseudo-random numbers with mean zero and variance σ^2. The variables are standardized so that $X'X$ and $X'y$ are in correlation form. We choose four different values for ρ as $\rho = 0.8, 0.99, 0.999$ and three different values for σ are investigated in this study, which are $\sigma = 0.5, 1$ and 1.5. For each choice of ρ, σ and n the experiment is replicated 1000 times and the *mse* is calculated as follows:

$$mse(\hat{\gamma}) = \frac{1}{1000} \sum_{j=1}^{1000} \sum_{i=1}^{5} (\hat{\gamma}_{ij} - \gamma_i)^2$$

where $\hat{\gamma}_{ij}$ denotes the estimate of the *i*-th parameter in *j*-th replication and γ_i, $i = 1, \ldots, p$ are the true parameter values. The results of the simulation are presented in Table 9.

Table 10. Scalar MSE of the estimators

n	ρ	$\hat{\beta}_{NL}$	$\hat{\beta}_{SNL}$
		$\sigma = 0.5$	
30	0.8	0.2047	0.2049
	0.99	0.3989	**0.3813**
	0.999	1.6660	**0.9926**
		$\sigma = 1$	
30	0.8	0.3574	**0.3461**
	0.99	2.3110	**1.2698**
	0.999	21.5594	**9.2984**
		$\sigma = 1.5$	
30	0.8	0.7406	**0.6249**
	0.99	6.5434	**3.0412**
	0.999	60.5396	**25.4627**
		$\sigma = 0.5$	
100	0.8	0.2285	0.2286
	0.99	0.3283	**0.3270**
	0.999	0.5994	**0.4995**
		$\sigma = 1$	
100	0.8	**0.3003**	**0.3003**
	0.99	0.7904	**0.6059**
	0.999	6.1964	**2.9095**
		$\sigma = 1.5$	
100	0.8	0.4891	**0.4794**
	0.99	2.3160	**1.2904**
	0.999	20.0570	**8.7287**

The increase in the σ or ρ, leads to an increase in the *mse* for all the estimators. For $\sigma = 0.5$ and weak collinearity, *mse* of NL estimator is smaller than *mse* of proposed estimator, however it may be noted that *mse* of proposed estimator superior to *mse* of NL estimator under strong and severely collinearity cases. The results indicate that this estimator is well behaved under multicollinearity conditions.

7. CONCLUSION

In order to overcome the multicollinearity problem in the linear regression model, this paper has proposed a shifted Liu-type estimator, which is based on the function of the biasing parameter k. Furthermore, this study has shown that proposed estimator has general applications and includes the OLS estimator, ridge estimator, Liu estimator and some estimators with two biasing parameters. The properties of this estimator have been discussed, together with the conditions under which it exhibits superior mean squared error matrix performance. Finally, these findings have been illustrated by a numerical example and a simulation study. The SNL estimator has displayed superior for the HK and HKB biasing parameters, while the estimators for the optimal biasing parameters have shown the same performance.

ACKNOWLEDGEMENT

No funding was received from any financial organization to conduct this research.

CONFLICT OF INTEREST

The authors declare that they have no any known financial or non-financial competing interests in any material discussed in this paper.

REFERENCES

[1] Guilkey D.K. & Murphy J.L., 1975. Directed Ridge Regression Techniques in Cases of Multicollinearity. *Journal of the American Statistical Association*, Vol. 70, No. 352, 769 – 775.

[2] Gunst R.F. & Mason R.L., 1977. Biased Estimation in Regression: An Evaluation Using Mean Squared Error. *Journal of the American Statistical Association*, Vol. 72, No. 359, 616 – 628.

[3] Hoerl A.E. & Kennard R.W., 1970a. Ridge Regression: Biased Estimation for Nonorthogonal Problems. *Technometrics*, Vol. 12, No. 1, 55 – 67.

[4] Hoerl A.E. & Kennard R.W., 1970b. Ridge Regression: Application to Non-Orthogonal Problems. *Technometrics*, Vol. 12, 69 – 82.

[5] Hoerl A.E., Kennard R.W. & Baldwin K.F., 1975. Ridge Regression: Some Simulations. *Communications in Statistics – Simulation and Computation*, Vol. 4, 105 – 123.

[6] Kurnaz F.S. & Akay K.U., 2015. A New Liu-Type Estimator. *Statistical Papers*, Vol. 56, No. 2, 495 – 517.

[7] Liu K., 1993. A New Class of Biased Estimate in Linear Regression. *Communications in Statistics - Theory and Methods*, Vol. 22, No. 2, 393 – 402.

[8] Liu K., 2003. Using Liu-Type Estimator to Combat Collinearity. *Communications in Statistics - Theory and Methods* Vol. 32, 1009 – 1020.

[9] McDonald G.C. & Galarneau D.I., 1975. A Monte Carlo Evaluation of Some Ridge-Type Estimators. *J Am Stat Assoc*, Vol. 70, 407 – 416.

[10] Özkale M.R. & Kaçiranlar S., 2007. The Restricted and Unrestricted Two-Parameter Estimators. *Communications in Statistics - Theory and Methods*, Vol. 36, 2707 – 2725.

[11] Sakallıoğlu S. & Kaçıranlar S., 2008. A New Biased Estimator Based on Ridge Estimation. *Statistical Papers*, Vol. 49, 669 – 689.

[12] Stein C., 1956. Inadmissibility of The Usual Estimator for Mean of Multivariate Normal Distribution. *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 1, 197 – 206.

[13] Trenkler G. & Toutenburg H., 1990. Mean Squared Error Matrix Comparisons Between Biased Estimators: An Overview of Recent Results. *Statistical Papers*, Vol. 31, 165 – 179.

[14] Vinod H.D., 1978. A Survey of Ridge Regression and Related Techniques for Improvements Over Ordinary Least Squares. *The Review of Economics and Statistics*, 121 – 131.

[15] Woods H, Stenior H.H. & Starke H.R., 1932. Effect of Composition of Portland Cement on Heat Evolved During Hardening. *Ind Eng Chem*, Vol. 24, 1207 – 1214.

[16] Yang H. & Chang X., 2010. A New Two-Parameter Estimator in Linear Regression. *Communications in Statistics - Theory and Methods*, Vol. 39, No. 6, 923 – 934.