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Small Stars versus Large Wheels

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Abstract. For given graphs G and H, the Ramsey number $R(G, H)$ is the smallest natural number n such that for every graph F of order n: either F contains G or the complement of F contains H. This paper investigates the Ramsey number $R(S_n, W_m)$ of small stars versus large wheels. We show that $R(S_6, W_8) = 14$. Furthermore, for $m \geq 2n-2$ and $n \geq 3$, then $R(S_n, W_m) = m + n - \mu$, where $\mu = 2$ if n is odd and m is even, and for otherwise $\mu = 1$.

Keywords : Ramsey numbers, stars, wheels

1 Introduction

For given graphs G and H, the *Ramsey number* $R(G, H)$ is defined as the smallest positive integer n such that for any graph F of order n, either F contains G or $\overline{F}$ contains H, where $\overline{F}$ is the complement of F. Chvátal and Harary [4] established a useful lower bound for finding the exact Ramsey numbers $R(G, H)$, namely $R(G, H) \geq (\chi(G) - 1)(C(H) - 1) + 1$, where $\chi(G)$ is the chromatic number of G and $C(H)$ is the number of vertices of the largest component of H. Since then the Ramsey numbers $R(G, H)$ for many combinations of graphs G and H have been extensively studied by various authors, see a nice survey paper [6]. In particular, the Ramsey numbers for combinations involving stars have also been investigated. Let S_n be a star of n vertices and W_m a wheel with m spokes. Surahmat et al. [7] proved that $R(S_n, W_4) = 2n - 1$ for $n \geq 3$ odd, otherwise $R(S_n, W_4) = 2n + 1$. They also showed $R(S_n, W_5) = 3n - 2$ for $n \geq 3$. Furthermore, it has been shown that if m is odd, $m \geq 5$ and $n \geq 2m - 4$, then $R(S_n, W_m) = 3n - 2$. This result is strengthened by Chen et al. In [3] by showing that

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this Ramsey number remains the same, even if m (≥ 5) is odd and $n \geq m-1 \geq 2$.

In this paper, we determine the Ramsey numbers $R(S_n, W_m)$ for open cases of n and m. The main results of this paper are the following.

Theorem 1. $R(S_6, W_8) = 14$.

Theorem 2. *For $m \geq 2n-2$ and $n \geq 3$, then $R(S_n, W_m) = m + n - \mu$, where $\mu = 2$ if n is odd and m is even, and otherwise $\mu = 1$.*

Before proving the theorems let us present some notations used in this note. Let $G(V, E)$ be a graph. Let $c(G)$ be the *circumference* of G, that is, the length of a longest cycle, and $g(G)$ be the *girth*, that is, the length of a shortest cycle. For any vertex $v \in V(G)$, the *neighborhood* $N(v)$ is the set of vertices adjacent to v in G, $N[v] = N(v) \cup \{v\}$. The number of vertices of a graph G is its *order*, written as $|G|$ and the degree of a vertex v in G is denoted by $d_G(v)$. The minimum (maximum) degree in G is denoted by $\delta(G)$ $(\Delta(G))$. For $S \subseteq V(G)$, $G[S]$ represents the subgraph induced by S in G. A graph on n vertices is *pancyclic* if it contains all cycles of every length l, $3 \leq l \leq n$. A graph is *weakly pancyclic* if it contains cycles of length from the girth to the circumference.

2 Some Lemmas

The following lemmas will be useful in proving our results.

Lemma 1. *(Bondy [1]). Let G be a graph of order n. If $\delta(G) \geq \frac{n}{2}$, then either G is pancyclic or n is even and $G = K_{\frac{n}{2},\frac{n}{2}}$.*

Lemma 2. *(Brandt et al. [2]). Every non-bipartite graph G with $\delta(G) \geq \frac{n+2}{3}$ is weakly pancyclic and has girth 3 or 4.*

Lemma 3. *(Dirac [5]). Let G be a 2-connected graph of order $n \geq 3$ with $\delta(G) = \delta$. Then $c(G) \geq \min\{2\delta, n\}$.*

3 The Proofs of Theorems

Proof of Theorem 1. Let F be a graph of order 14. Suppose F contains no S_6, and so $d_F(x) \leq 4\ \forall x \in F$. Let there exist $x_0 \in F, d_F(x_0) \leq 3$. If $A = V(F)\backslash N[x_0]$ and $T = F[A]$ then $|T| \geq 10$ and $\delta(\overline{T}) \geq |T| - 5) \geq \frac{|\overline{T}|}{2}$. By Lemma 1, $\overline{T}$ contains a C_8. With the center x_0, we obtain wheel W_8 in $\overline{F}$. Now, let for each $v \in F$, $d_F(v) = 4$. If $A = V(F)\backslash N[v_0]$ where v_0 any vertex of F , $T = F[A]$, then $|T| = 9$. Observe that $d_{\overline{T}}(v) = |T| - 5 = 4 \geq \frac{|\overline{T}|+2}{3}$. Since $|\overline{T}| = 9$ and $d_{\overline{T}}(v) = 4$, $\forall v \in \overline{T}$, obviously $\overline{T}$ is connected and non bipartite. Hence, $\kappa(\overline{T}) > 0$. Since, $d_{\overline{T}}(v) \geq \frac{|\overline{T}|+2}{3}$ and $\overline{T}$ is non bipartite, then by Lemma 2, $\overline{T}$ is weakly *pancyclic*, and has girth 3 or 4. In other words, $\bar{(T)}$ contains all cycles C_m, with $g(\overline{T}) \leq m \leq c(\overline{T})$, where $g(\overline{T}) = 3$ or 4 and $c(\overline{T})$ is the length of its largest cycle. Next, we will to find out $c(\overline{T})$.

Let $\kappa(\overline{T}) = 1$, say u_0 is a cut-vertex, then it is easy to see that $\overline{T} = u_0 + 2K_4$. This is impossible, contradict with $d(v) = 4$, $\forall v \in \overline{T}$. Hence, $\kappa(\overline{T}) \geq 2$. Thus, $\overline{T}$ is 2-connected. By Lemma 3, $c(\overline{T}) \geq \min\{2(4), 9\}$. Therefore, $\overline{F}$ contains W_8, with the center v_0, and so $R(S_6, W_8) \leq 14$.

On the other hand, it is not difficult to see that graph $F_1 = K_{4,4} \cup K_5$ contain no S_6 and its complement contain no W_8. Observe that F_1 has 13 vertices. Hence, we have $R(S_6, W_8) \geq 14$. □

Proof of Theorem 2.

For $m \geq 2n - 2$ and $n \geq 4$. Let n is odd and m is even. Since $(n-2) - \textit{regular regular}$ with the order $m + n - 3$ contain no S_n and its complement contain no W_m, then $R(S_n, W_m) \geq m + n - 2$. On the other hand, let F be a graph of order $m + n - 2$. Suppose F contains no S_n, and so $d_F(v) \leq n - 2$, $\forall v \in F$. Since n is odd and m is even, then there exists $x_0 \in F$ with $d_F(x_0) \leq n - 3$. Let $A = V(F)\backslash N[x_0]$, and $T = F[A]$. Since for each $v \in T$, $d_T(v) \leq n-2$ and $|T| \geq m$, then $d_{\overline{T}}(v) \geq |T| - (n-1) \geq \frac{|\overline{T}|}{2}$. This implies that $\overline{T}$ contains a C_m (by Lemma 1). Hence, $\overline{F}$ contains a W_m, with the center x_0. Therefore, $R(S_n, W_m) \leq m + n - 2$ for odd n and even m.

Now, for other n and m, consider $(n-2)-regular\ graph$ with order $m+n-2$, call F_1. We can verify that F contain no S_n and its complement contain no W_m, Hence, we have $R(S_n, W_m) \geq m+n-1$. On the other hand, let F be a graph of order $m+n-1$. Suppose F contains no S_n, and so $d_F(v) \leq n-2$, $\forall v \in F$. If $B = V(F) \backslash N[v_0]$, and $T = F[B]$, then $|T| \geq m$. Since for each $v \in T$, and $d_T(v) \leq n-2$, then $d_{\overline{T}}(v) \geq |T| - (n-1) \geq \frac{|\overline{T}|}{2}$. By Lemma 1, $\overline{T}$ contains a cycle C_l, where $3 \leq l \leq m \leq |\overline{T}|$.

Therefore, we obtain a wheel W_m in $\overline{F}$, with the center v_0. Hence, $R(S_n, W_m) \leq m+n-1$. □

4 Open Problems

As a final remark, let us present the following open problem to work on.

Problem 1. Find the Ramsey number $R(S_n, W_m)$ for $n \geq 4$ and all m, $n+1 \leq m < 2n-2$.

Problem 2. Find the Ramsey number $R(S_{n,r}, W_m)$ for any n, r and m.

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