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Metric Dimension of Second Order Complete Graph with Honeycomb Network Cross Operation Product

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Abstract
Metric dimension is a topic in graph theory that has been developed in terms of the concept and its application. Let G be a connected graph and S be a vertex subset on G. The set S is called a resolving set for G if every vertex on G has a distinct representation of one to each other of S. A resolving set containing a minimum cardinality is called basis. The metric dimension on G is cardinality of basis on G, notated with dim (G). In this case, the cross-product graph will be used for the research. The aim of this research is to determine the metric dimension of the second order complete graph (K_2) with honeycomb network $(\mathrm{HC(n)})$ cross-operation product. Utilizing direct proofing, we generated $d\ (\mathrm{K}_2 \times \mathrm{HC(n)}) = 3$.

Keywords: metric dimension, cross product graph, complete graph, honeycomb network

1. INTRODUCTION AND PRELIMINARIES

Graph is a pair of sets, the first of its element is called a vertex, while the other of its element is a pair of vertices, which are called an edge [14]. Several classes of simple graphs have distinctive features such as paths, cycles, wheels, complete graphs and others. A complete graph is a special graph in which every pair of its vertices is adjacent [6]. Complete graph, which has n vertices notated with K_n. One of special graph classes for instance is honeycomb network. According to [12], $n-$dimension honeycomb network denoted by $HC(n)$, wherein n tells the number of hexagons from centre hexagon to the most surface one. Let u and v be the vertices in connected graph G, the distance between u and v on graph G is the minimum length of a path between u and v on G denoted by $d(u, v)$. Representation of a vertex v towards set S in graph G is the distance of vertex v to each vertex in S, denoted by $r(v|S)$.

One topic of the study in graph theory is metric dimension. Metric dimension in graph theory firstly occurred at 1975, specifically written in [5] and [16]. For theoretical purposes, the metric dimension provides insights into graph structure, aiding in understanding vertex distinguishability, classification of graph families, and contributing to problems in combinatorial optimization and graph algorithms. In terms to obtain the metric dimension of a certain graph form, formerly required subclasses analysis in order to facilitate metric dimension investigation in general [11]. Let u and

v be the vertices in connected graph G with $S \subset V(G)$. If the representation of $r(u|S) \neq r(v|S)$, then set S is called a resolving set. Minimum cardinality of a resolving set S on graph G labelled as metric dimension of graph G denoted by $dim\ (G)$ and S is called a basis of G [17].

Discussion on metric dimension of special graph had been studied by several graph theory researchers. [21] characterized the graphs for which the edge metric dimension of graphs is n − 1. Recently, [18] gave the characterization of all connected bipartite graphs with edge metric dimension n − 2, which partially answers an open problem of [21]. They also investigated the relationship between the edge metric dimension and the clique number of a graph G. In the year of 2000, [2] acquired $dim\ (K_n)$ is $n-1$ for $n \geq 3$, $dim\ (C_n)$ is 2 for $n\ \geq 3$ and $dim\ (G) = n-2$ if only if $G = K_{r,s}\ (r,s \geq 1)$, $G = K_r + \overline{K_s}\ (r \geq 1, s \geq 2)$ or $G = K_r + (K_1 \cup K_s)\ (r,s \geq 1)$ for $r + s = n$. In the year of 2005, [10] found these results: $dim\ (P_m \times P_n) = 2$, $dim\ (P_m \times K_n) = n-1$ for $n \geq 3$, $dim\ (W_{1,n}) = \left\lfloor \frac{2n+2}{5} \right\rfloor$ for $n \notin \{3,6\}$ where $W_{1,n}$ is a wheel graph with an order $n+1$, $dim\ \ (F_{1,n}) = \left\lfloor \frac{2n+2}{5} \right\rfloor$ for $n \notin \{1,2,3,6\}$ where $F_{1,n}$ is a fan graph with an order $n+1$. The other results shows that $dim\ (P_m \times C_n) = \begin{cases} 2, \text{if } n \text{ is odd;} \\ 3, \text{if } n \text{ is even}, m \neq 1, \end{cases}$ and $dim\ (C_m \times C_n) = \begin{cases} 4, \text{if } m,n \text{ are even;} \\ 3, \text{otherwise.} \end{cases}$ Metric dimension of complete graph with cycle graph cross operation product can be presented as follows.

Theorem 1.1 For $m \geq 4$, $dim\ \ (K_m \times C_n) = \begin{cases} m, \ \text{if } m = 4 \text{ and } n \text{ is odd;} \\ m-1, \text{otherwise.} \end{cases}$

Theorem 1.2 For $m \leq n$,

$$dim\ \ (K_m \times K_n) = \begin{cases} n-1, \ \text{if } \ 2m-2 < \ n; \\ \left\lfloor \frac{2m+2n-2}{3} \right\rfloor, \text{if } \ 2m-2 \geq \ n. \end{cases}$$

[10] also stated that if graphs G and H are a simple graph, where $|G|$ and $|H|$ represent each of the cardinality, then:

$max\{dim\ (G), dim(H)\} \leq dim(G \times H) \leq min\{dim(G) + |H|, dim(H) + |G|\} - 1$,

$2 \leq dim(G) \leq dim(H) \leq dim(G \times H) \leq min\{dim(G) + |H|, dim(H) + |G|\} - 2$,

$dim\ (G) + dim\ (H) \leq dim\ (G + H)\ , dim(G) \leq dim(G \times K_2) \leq dim(G) + 1$,

$dim(G) \leq dim(G \times P_n) \leq dim(G) + 1$, and

$dim(G \times C_n)\} \leq \begin{cases} dim\ (G) + 1, \text{if } n \text{ is odd;} \\ dim\ (G) + 2, \text{if } n \text{ is even.} \end{cases}$

$dim\ (G) \leq dim(G \times K_2) \leq dim(G) + 1$, $dim\ (G) \leq dim(G \times P_n) \leq dim(G) + 1$, and

$dim(G \times C_n)\} \leq \begin{cases} dim\ (G) + 1, \text{if } n \text{ is odd;} \\ dim\ (G) + 2, \text{if } n \text{ is even.} \end{cases}$

Another finding of [10] is if $G = T + e$ be a graph with one cycle, then $dim\ (T) - 2 \leq dim(T + e) \leq dim\ (T) + 1$.[9] in their paper summarized the results as follows:

(i) Let G be a connected graph and H be a graph with order at least 2. Then:
$dim\ (G \odot H) = \begin{cases} |G|\ dim\ (H)\ , \text{if } H \text{ contains a dominant vertex;} \\ |G|\ dim\ (K_1 + H), \text{otherwise.} \end{cases}$

(ii) Let G and H be a connected graph, then $dim(K_1 + H) \geq dim\ (H) + 1$.

(iii) $dim\ (G \odot K_n) = |G|\ (n-1), n \geq 2$.

(iv) $dim\ (G \odot H) = |G| \left\lfloor \frac{2n+2}{5} \right\rfloor$, for $H \in \{C_n, P_n\}$ and C_n, P_n do not contain a dominant vertex and $dim\ (K_1 + H) = \left\lfloor \frac{2n+2}{5} \right\rfloor$.

Some other interesting results on the metric dimension and references can be found in [19] and [20]. In accordance with [2] finding, stated that $dim\,(S_n) = n - 2$, so that $dim\,(G \odot S_n) = |G|(\mathrm{n} - 2)$. [13] also obtained $dim\,(HC(n))$ is 3 and if follows that [10] found $dim\,(G) \leq dim(G \times K_2) \leq dim(G) + 1$, such that $dim\,(HC(n)) \leq dim\,(HC(n) \times K_2) \leq dim(HC(n)) + 1$. Thus, $3 \leq dim(HC(n)) \times K_2) \leq 4$. In this paper, it will be given the metric dimension of graph $HC(n) \times K_2$ scientifically, specifically it will be shown that $dim(HC(n) \times K_2) = 3$.

This research utilized literature study method using related and relevant references of scientific works, which are 3 books and 18 journals. The methodology will initiated with the construction of second order complete graph with honeycomb network cross operation product. The largest lower bound and the smallest upper bound of metric dimension for graph $K_2 \times HC(n)$ will then be determined afterward, leading to the final result, $dim(K_2 \times HC(n)) = 3$.

2. MAIN RESULTS

Various writer described their own unique definition of graph. According to [3] and [6], simple graph G is a pair $(V(G), E(G))$, with $V(G)$ is a finite and not empty discrete set, which its element is called vertex and $E(G)$ is called a set pair whose elements are unordered and distinct from $V(G)$, which is named edge. Mathematically, graph $G = (V(G), E(G))$ with $V(G) = \{u\colon u \text{ is called vertex}\}$ is called vertex set and $(E(G)) = \{(u, v)\colon u, v \in V(G)\}$, with (u, v) is called edge, lateral or line and $E(G)$ is called edge set. In the following discussion, (u, v) edge only be written as uv. For every empty graph denoted by (,), also can be written as { }. Graph with order 0 or 1 said to be trivial [4].

There are a number of graph classes, a few among them are complete graph, one of special graph, whose every both of its vertices are neighbour, which is denoted by K_n. As a consequence, each vertex in complete graph has the same degree. Honeycomb network built from a number of cycles C_6, denoted by $HC(n)$ with $n \geq 2$ is the layer order n within a honeycomb network. There are a number of operations of graph. One of it is cross operation. Defined in [8], let G be a graph with vertex set $V(G)$ and edge set $E(G)$, H be a graph with vertex set $V(H)$ and edge set $E(H)$, then: cross operation graph between G and H written as $G \times H$, be a graph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{xy\,|x = u_1v_1,\ y = u_2v_2;\ u_1 = u_2 \ and\ v_1v_2 \in E(H)\ or\ v_1 = v_2\ and\ u_1u_2 \in E(G)\}$. Metric dimension of graph G notated with $dim\,(G)$ is a minimum resolving set, which is called basis [1]. Furthermore, [15] given following statement.

Definition 2.1 *Let $S \subseteq V(G)$ and $S = \{v_1, v_2, v_3, \dots, v_k\}$. Vertex representation $r(v|S)$ of vertex v towards S presented as $[d(v, v_1), d(v, v_2), d(v, v_3), \dots, d(v, v_k)]$. Set $S \subseteq (G)$ is called a resolving set if $r(u|S) \neq r(v|S)$ for every vertex $u, v \in V(G)$ with $u \neq v$.*

Theorem 2.1 *Let G be a honeycomb network, which is denoted by $HC(n)$, then $dim\,(G) = 3$.* [13]

This paper wiil be discussing about the metric dimension graph K_2 and graph $HC(n)$ determination. Honeycomb network concept is explained below. Honeycomb network $HC(n)$ with $n \geq 2$ is the layer order n of honeycomb network, obtained with sticking an amount $6(n-1)$ of cycles C_6 on the outermost layer $HC(n-1)$. The total number of honeycomb network's vertices is $|V(HC(n)| = 6n^2$ and the total number of the edges is $|E(HC(n)| = 9n^2 - 3n$. So that honeycomb network $HC(n)$ is a graph with vertex set and edge given as following

$$V\big(HC(n)\big) = \{u_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq 12i - 6\} \quad (2.1)$$

$$
\begin{aligned}
E\big(HC(n)\big) = &\{(u_{i,j}, u_{i,j+1}) \mid 1 \le i \le n, 1 \le j \le 12i-6\} \cup \\
&\{(u_{1,j}, u_{2,3(j-1)}) \mid 2 \le j \le (12(i+1)-6)+1 \bmod 6\} \cup \\
&\{(u_{i,j}, u_{i+1,j+1}) \mid 2 \le i \le n, j = 2,4,6,\dots, j_1 \text{ with } j_1 = 2i \text{ and } |j| = i\} \cup \\
&\{(u_{i,j}, u_{i+1,j+3}) \mid 2 \le i \le n, j = j_1+1, j_1+3, j_1+5,\dots, j_2 \text{ and } |j| = i\} \cup \\
&\{(u_{i,j}, u_{i+1,j+5}) \mid 2 \le i \le n, j = j_2+1, j_2+3, j_2+5,\dots, j_3 \text{ and } |j| = i\} \cup \\
&\{(u_{i,j}, u_{i+1,j+7}) \mid 2 \le i \le n, j = j_3+1, j_3+3, j_3+5,\dots, j_4 \text{ and } |j| = i\} \cup \\
&\{(u_{i,j}, u_{i+1,j+9}) \mid 2 \le i \le n, j = j_4+1, j_4+3, j_4+5,\dots, j_5 \text{ and } |j| = i\} \cup \\
&\{(u_{i,j}, u_{i+1,j+11}) \mid 2 \le i \le n, j = j_5+1, j_5+3, j_5+5,\dots, j_6 \\
&\text{with } j_6 = (12i-6)+1 \bmod 6 \text{ and } |j| = i\} \qquad (2.2)
\end{aligned}
$$

$\text{with } j = 12i-6, j+1 = j+1 \bmod (12i-6).$

The largest lower bound of $dim\left(K_2 \times HC(n)\right)$ given by Carmen *et al*. (2005) presented in this following theorem.

Theorem 2.2 *For any graph G*, $dim(G) \le dim(K_2 \times G) \le dim(G) + 1$.

Based on vertex set dan edge set labelling of graph $HC(n)$, the labelling of vertex set and edge set of a graph $K_2 \times HC(n)$ presented as following with $au_{i,j}$ only be written with $u_{i,j}$ and $bu_{i,j}$ only be written with $u'_{i,j}$.

$$V\big(K_2 \times HC(n)\big) = \{u_{i,j} \mid 1 \le i \le n, 1 \le j \le 12i-6\} \cup \{u'_{i,j} \mid 1 \le i \le n, 1 \le j \le 12i-6\} \qquad (2.3)$$

$$
\begin{aligned}
&E\big(K_2 \times HC(n)\big) = \\
&\{(u_{i,j}, u_{i,j+1}) \mid 1 \le i \le n, 1 \le j \le 12i-6\} \cup \\
&\{(u_{1,j}, u_{2,3(j-1)}) \mid 2 \le j \le (12(i+1)-6)+1 \bmod 6\} \cup \\
&\{(u_{i,j}, u_{i+1,j+1}) \mid 2 \le i \le n, j = 2,4,6,\dots, j_1 \text{ with } j_1 = 2i \text{ and } |j| = i\} \cup \\
&\{(u_{i,j}, u_{i+1,j+3}) \mid 2 \le i \le n, j = j_1+1, j_1+3, j_1+5,\dots, j_2 \text{ and } |j| = i\} \cup \\
&\{(u_{i,j}, u_{i+1,j+5}) \mid 2 \le i \le n, j = j_2+1, j_2+3, j_2+5,\dots, j_3 \text{ and } |j| = i\} \cup \\
&\{(u_{i,j}, u_{i+1,j+7}) \mid 2 \le i \le n, j = j_3+1, j_3+3, j_3+5,\dots, j_4 \text{ and } |j| = i\} \cup \\
&\{(u_{i,j}, u_{i+1,j+9}) \mid 2 \le i \le n, j = j_4+1, j_4+3, j_4+5,\dots, j_5 \text{ and } |j| = i\} \cup \\
&\{(u_{i,j}, u_{i+1,j+11}) \mid 2 \le i \le n, j = j_5+1, j_5+3, j_5+5,\dots, j_6 \\
&\quad \text{with } j_6 = (12i-6)+1 \bmod 6 \text{ and } |j| = i\} \cup \\
&\{(u'_{i,j}, u'_{i,j+1}) \mid 1 \le i \le n, 1 \le j \le 12i-6\} \cup \\
&\{(u'_{1,j}, u'_{2,3(j-1)}) \mid 2 \le j \le (12(i+1)-6)+1 \bmod 6\} \cup \\
&\{(u'_{i,j}, u'_{i+1,j+1}) \mid 2 \le i \le n, j = 2,4,6,\dots, j_1 \text{ with } j_1 = 2i \text{ and } |j| = i\} \cup \\
&\{(u'_{i,j}, u'_{i+1,j+3}) \mid 2 \le i \le n, j = j_1+1, j_1+3, j_1+5,\dots, j_2 \text{ and } |j| = i\} \cup \\
&\{(u'_{i,j}, u'_{i+1,j+5}) \mid 2 \le i \le n, j = j_2+1, j_2+3, j_2+5,\dots, j_3 \text{ and } |j| = i\} \cup \\
&\{(u'_{i,j}, u'_{i+1,j+7}) \mid 2 \le i \le n, j = j_3+1, j_3+3, j_3+5,\dots, j_4 \text{ and } |j| = i\} \cup \\
&\{(u'_{i,j}, u'_{i+1,j+9}) \mid 2 \le i \le n, j = j_4+1, j_4+3, j_4+5,\dots, j_5 \text{ and } |j| = i\} \cup \\
&\{(u'_{i,j}, u'_{i+1,j+11}) \mid 2 \le i \le n, j = j_5+1, j_5+3, j_5+5,\dots, j_6 \\
&\quad \text{with } j_6 = (12i-6)+1 \bmod 6 \text{ and } |j| = i\} \cup \\
&\{(u_{i,j}, u'_{i,j}) \mid 1 \le i \le n, 1 \le j \le 12n-6\} \qquad (2.4) \\
&\quad \text{with } j = 12i-6, j+1 = j+1 \bmod (12i-6).
\end{aligned}
$$

Based on vertex set and edge set of $K_2 \times HC(n)$ definition, for $1 \leq i,k \leq n$ and $1 \leq j,l \leq 12i-6$, obtained some characteristic related to vertices distance on graph $K_2 \times HC(n)$ as follows.

1. $$d(u'_{i,j}, u'_{k,l}) = \begin{cases} 0, & for\ i = k\ and\ j = l \\ 1, & for\ i = k\ and\ j = l-1\ or\ j = 12i-6 \\ & \quad or\ i = k-1\ and\ j = 12i-6 \\ 2, & for\ i = k\ and\ j = l-2\ or\ j = l-4i^2 \\ & \quad or\ i = k-1\ and\ j = l\ or\ j = l-2, \\ & \quad or\ j = 12i-9\ or\ j = 12i-7 \\ & \vdots \\ 4i-1, & for\ i = k = n\ and\ l = \frac{12i-4}{2}\ for\ j = 1, \\ & \quad l = \frac{12i-4}{2}\ or\ l = \frac{12i-8}{2}\ for\ j > 1\ (odd), \\ & \quad l = \frac{12i-2}{2}\ or\ l = \frac{12i+2}{2}\ for\ j \geq 2\ (even) \end{cases}$$

2. $$d(u'_{i,j}, u'_{k,l}) = \begin{cases} 0, & for\ i = k\ and\ j = l \\ 1, & for\ i = k\ and\ j = l-1\ or\ j = 12i-6 \\ & \quad or\ i = k-1\ and\ j = 12i-6 \\ 2, & for\ i = k\ and\ j = l-2\ or\ j = l-4i^2 \\ & \quad or\ i = k-1\ and\ j = l\ or\ j = l-2, \\ & \quad or\ j = 12i-9\ or\ j = 12i-7 \\ & \vdots \\ 4i-1, & for\ i = k = n\ and\ l = \frac{12i-4}{2}\ for\ j = 1, \\ & \quad l = \frac{12i-4}{2}\ or\ l = \frac{12i-8}{2}\ for\ j > 1\ (odd), \\ & \quad l = \frac{12i-2}{2}\ or\ l = \frac{12i+2}{2}\ for\ j \geq 2\ (even) \end{cases}$$

3. $$d(u_{i,j}, u'_{k,l}) = \begin{cases} 0, & for\ i = k\ and\ j = l \\ 1, & for\ i = k\ and\ j = l-1\ or\ j = 12i-6 \\ & \quad or\ i = k-1\ and\ j = 12i-6 \\ 2, & for\ i = k\ and\ j = l-2\ or\ j = l-4i^2 \\ & \quad or\ i = k-1\ and\ j = l\ or\ j = l-2, \\ & \quad or\ j = 12i-9\ or\ j = 12i-7 \\ & \vdots \\ 4i, & for\ i = k = n\ and\ l = \frac{12i-4}{2}\ for\ j = 1, \\ & \quad l = \frac{12i-4}{2}\ or\ l = \frac{12i-8}{2}\ for\ j > 1\ (odd), \\ & \quad l = \frac{12i-2}{2}\ or\ l = \frac{12i+2}{2}\ for\ j \geq 2\ (even) \end{cases}$$

In a forward discussion, there will be an explanation on metric dimension of graph $K_2 \times HC(n)$ for $n \in Z^+$. To investigate the metric dimension on graph $K_2 \times HC(n)$, initially determined the largest lower bound and the smallest upper bound on the metric dimension of the graph. Metric dimension is requiring resolving set W with minimum cardinality. Based on Theorem 2.2, we obtained the largest lower bound of $K_2 \times HC(n)$ is $3 \leq dim\left(K_2 \times HC(n)\right) \leq 4$. Given the main result of the research shown in these following propositions.

Proposition 2.1 *Given graph $K_2 \times HC(3)$, then the metric dimension of graph $K_2\ X\ HC(3)$ is $dim\left(K_2 \times HC(3)\right) = 3$.*

Proof:
According to Equation 2.3 and 2.4, we obtained the vertex set and the edge set of a graph $K_2 \times HC(3)$ as follows.

$$V\big(K_2 \times HC(3)\big) = \{u_{1,j} \mid 1 \le j \le 6\} \cup \{u_{2,j} \mid 1 \le j \le 18\} \cup \{u_{3,j} \mid 1 \le j \le 30\} \cup \{u'_{1,j} \mid 1 \le j \le 6\} \cup \{u'_{2,j} \mid 1 \le j \le 18\} \cup \{u'_{3,j} \mid 1 \le j \le 30\}$$

$$\begin{aligned} E\big(K_2 \times HC(3)\big) = {} & \{(u_{1,j}, u_{1,j+1}) \mid 1 \le j \le 6\} \cup \{(u_{2,j}, u_{2,j+1}) \mid 1 \le j \le 18\} \cup \\ & \{(u_{3,j}, u_{3,j+1}) \mid 1 \le j \le 30\} \cup \{(u_{1,j}, u_{2,3(j-1)}) \mid 2 \le j \le 6\} \cup \\ & \{(u_{2,j}, u_{3,j+1}) \mid j = 2,4\} \cup \{(u_{2,j}, u_{3,j+3}) \mid j = 5,7\} \cup \\ & \{(u_{2,j}, u_{3,j+5}) \mid j = 8,10\} \cup \{(u_{2,j}, u_{3,j+7}) \mid j = 11,13\} \cup \\ & \{(u_{2,j}, u_{3,j+9}) \mid j = 14,16\} \cup \{(u_{2,j}, u_{3,j+11}) \mid j = 17,19\} \cup \\ & \{(u'_{1,j}, u'_{1,j+1}) \mid 1 \le j \le 6\} \cup \{(u'_{2,j}, u'_{2,j+1}) \mid 1 \le j \le 18\} \cup \\ & \{(u'_{3,j}, u'_{3,j+1}) \mid 1 \le j \le 30\} \cup \{(u'_{1,j}, u'_{2,3(j-1)}) \mid 2 \le j \le 6\} \cup \\ & \{(u'_{2,j}, u'_{3,j+1}) \mid j = 2,4\} \cup \{(u'_{2,j}, u'_{3,j+3}) \mid j = 5,7\} \cup \\ & \{(u'_{2,j}, u'_{3,j+5}) \mid j = 8,10\} \cup \{(u'_{2,j}, u'_{3,j+7}) \mid j = 11,13\} \cup \\ & \{(u'_{2,j}, u'_{3,j+9}) \mid j = 14,16\} \cup \{(u'_{2,j}, u'_{3,j+11}) \mid j = 17,19\} \cup \\ & \{(u_{1,j}, u'_{1,j}) \mid 1 \le j \le 6\} \cup \{(u_{2,j}, u'_{2,j}) \mid 1 \le j \le 18\} \cup \\ & \{(u_{3,j}, u'_{3,j}) \mid 1 \le j \le 30\} \end{aligned}$$

$with\ 7\ mod\ 6 = 19\ mod\ 6 = 31\ mod\ 6 = 1.$

Figure of graph $K_2 \times HC(3)$ could be shown at Figure 2.1.

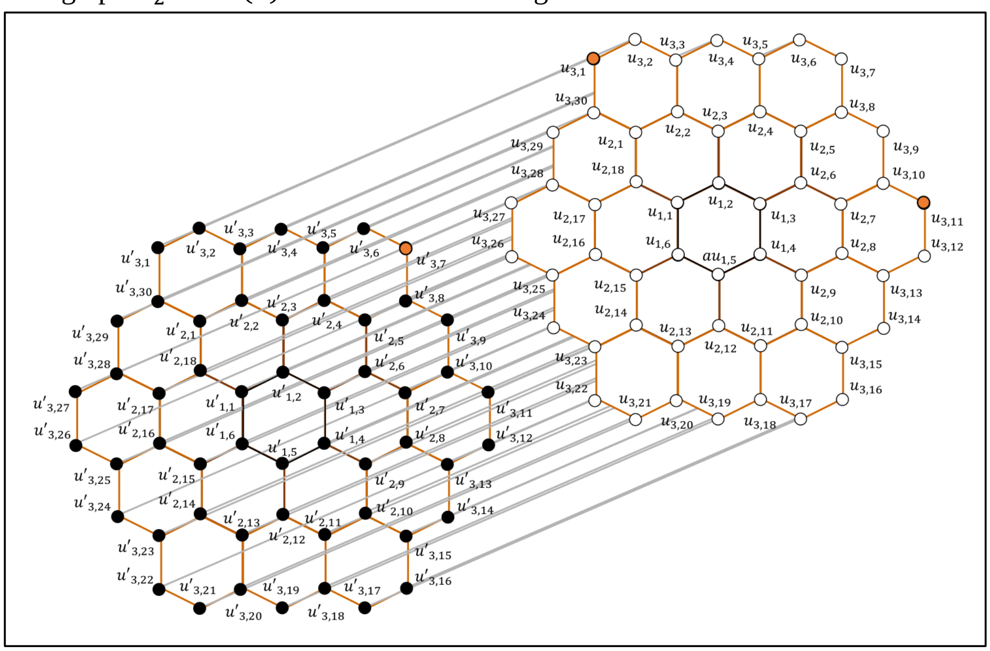

Figure 2.1. Graph K_2 X $HC(n)$

Refer to Theorem 2.1, which is stated that $dim\,(HC(n)) = 3$, with $HC(n) \subseteq K_2 \times HC(n)$ and Theorem 2.2, which is stated that $dim(G) \le dim(G \times K_2) \le dim(G) + 1$, then its largest lower bound is $dim\,(K_2 \times HC(3)) \ge 3$. In addition, it will be shown that its smallest upper bound is $dim\,(K_2 \times HC(3)) \le 3$. Choose the subset $V(K_2 \times HC(3))$, write $W = \{u_{3,1}, u'_{3,7}, u_{3,11}\}$ so that we obtained the representations for every vertex on graph $K_2 \times HC(3)$ toward W given as follows.

j	$r(u_{i,j}\|W)$	$r(u'_{i,j}\|W)$
1 2 3	$r(u_{1,j}\|W) = [2n-3+j, 2n+2-j, 2n+1-j]$,	$r(u'_{1,j}\|W) = [2n-2+j, 2n+1-j, 2n+2-j]$,
4 5	$r(u_{1,j}\|W) = [2n+5-j, 2n-4+j, 2n-5+j]$,	$r(u'_{1,j}\|W) = [2n+6-j, 2n-5+j, 2n-4+j]$,
6	$r(u_{1,j}\|W) = [2n-5+j, 2n+8-j, 2n-5+j]$,	$r(u'_{1,j}\|W) = [2n-4+j, 2n+7-j, 2n-4+j]$,
1 2 3 4 5	$r(u_{n-1,j}\|W) = [j+1, 2n+2-j, 4n-3-j]$,	$r(u'_{n-1,j}\|W) = [j+2, 2n+1-j, 4n-2-j]$,
6 7	$r(u_{n-1,j}\|W) = [j+1, 2n-8+j, 4n-3-j]$,	$r(u'_{n-1,j}\|W) = [j+2, 2n-9+j, 4n-2-j]$,
8 9	$r(u_{n-1,j}\|W) = [17-j, 2n-8+j, 4n-17+j]$,	$r(u'_{n-1,j}\|W) = [18-j, 2n-9+j, 4n-16+j]$,
10 11	$r(u_{n-1,j}\|W) = [19-j, 2n-8+j, 4n-17+j]$,	$r(u'_{n-1,j}\|W) = [20-j, 2n-9+j, 4n-16+j]$,
12 13	$r(u_{n-1,j}\|W) = [j-5, 2n-10+j, 4n-17+j]$,	$r(u'_{n-1,j}\|W) = [j-4, 2n-11+j, 4n-16+j]$,
14 15	$r(u_{n-1,j}\|W) = [21-j, 2n+18-j, 4n+11-j]$,	$r(u'_{n-1,j}\|W) = [22-j, 2n+17-j, 4n+12-j]$,
16 17 18	$r(u_{n-1,j}\|W) = [21-j, 2n+20-j, 4n+13-j]$,	$r(u'_{n-1,j}\|W) = [22-j, 2n+19-j, 4n+14-j]$,
1 2 3 4 5 6 7	$r(u_{n,j}\|W) = [j-1, 2n+2-j, 4n-1-j]$,	$r(u'_{n,j}\|W) = [j, 2n+1-j, 4n-j]$,
8 9 10 11	$r(u_{n,j}\|W) = [j-1, 2n-12+j, 4n-1-j]$,	$r(u'_{n,j}\|W) = [j, 2n-13+j, 4n-j]$,
12 13	$r(u_{n,j}\|W) = [j-1, 2n-12+j, 4n-23+j]$,	$r(u'_{n,j}\|W) = [j, 2n-13+j, 4n-22+j]$,
14 15	$r(u_{n,j}\|W) = [22-j, 2n-12+j, 4n-23+j]$,	$r(u'_{n,j}\|W) = [23-j, 2n-13+j, 4n-22+j]$,
16 17	$r(u_{n,j}\|W) = [24-j, 2n-12+j, 4n-23+j]$,	$r(u'_{n,j}\|W) = [25-j, 2n-13+j, 4n-22+j]$,
18 19	$r(u_{n,j}\|W) = [j-12, 2n-14+j, 4n-23+j]$,	$r(u'_{n,j}\|W) = [j-11, 2n-15+j, 4n-22+j]$,
20 21	$r(u_{n,j}\|W) = [j-11, 2n-16+j, 4n-23+j]$,	$r(u'_{n,j}\|W) = [j-10, 2n-17+j, 4n-22+j]$,
22 23	$r(u_{n,j}\|W) = [31-j, 2n+28-j, 4n+21-j]$,	$r(u'_{n,j}\|W) = [32-j, 2n+27-j, 4n+22-j]$,
24 25	$r(u_{n,j}\|W) = [31-j, 2n+30-j, 4n+23-j]$,	$r(u'_{n,j}\|W) = [32-j, 2n+29-j, 4n+24-j]$,
26 27 28	$r(u_{n,j}\|W) = [31-j, 2n+32-j, 4n+25-j]$,	$r(u'_{n,j}\|W) = [32-j, 2n+31-j, 4n+26-j]$,
29 30	$r(u_{n,j}\|W) = [31-j, 2n+32-j, 4n+27-j]$,	$r(u'_{n,j}\|W) = [32-j, 2n+31-j, 4n+28-j]$.

From the vertex representation above, we obtained that W is the resolving set of a graph $K_2 \times HC(3)$ since $r(v_n|W) \neq r(v_m|W)$ for $\forall n, m \in (K_2 \times HC(3))$ and $n \neq m$. So that, the smallest upper bound of graph $K_2 \times HC(3)$ is $dim(K_2 \times HC(3)) \leq 3$. Through this analysis, we obtained the largest lower bound and the smallest upper bound on the metric dimension of graph $K_2 \times HC(3)$ is $3 \leq dim\ (K_2 \times HC(3)) \leq 3$. Thus, $dim\ (K_2 \times HC(3)) = 3$. ■

Proposition 3.2 *Given graph* $K_2 \times HC(4)$, *then the metric dimension of graph* $K_2 \times HC(4)$ *is* $dim\left(K_2 \times HC(4)\right) = 3$.
Proof:
According to Equation 2.3 and 2.4, we obtained the vertex set and the edge set of a graph $K_2 \times HC(4)$ as follows.

$$V\left(K_2 \times HC(4)\right) = \left\{u_{1,j} \mid 1 \leq j \leq 6\right\} \cup \left\{u_{2,j} \mid 1 \leq j \leq 18\right\} \cup \left\{u_{3,j} \mid 1 \leq j \leq 30\right\} \cup \left\{u_{4,j} \mid 1 \leq j \leq 42\right\} \cup \left\{u'_{1,j} \mid 1 \leq j \leq 6\right\} \cup \left\{u'_{2,j} \mid 1 \leq j \leq 18\right\} \cup \left\{u'_{3,j} \mid 1 \leq j \leq 30\right\} \cup \left\{u'_{4,j} \mid 1 \leq j \leq 42\right\}$$

$$\begin{aligned} E\left(K_2 \times HC(4)\right) = & \left\{\left(u_{1,j}, u_{1,j+1}\right) \middle| 1 \leq j \leq 6\right\} \cup \left\{\left(u_{2,j}, u_{2,j+1}\right) \middle| 1 \leq j \leq 18\right\} \cup \\ & \left\{\left(u_{3,j}, u_{3,j+1}\right) \middle| 1 \leq j \leq 30\right\} \cup \left\{\left(u_{4,j}, u_{4,j+1}\right) \middle| 1 \leq j \leq 42\right\} \cup \\ & \left\{\left(u_{1,j}, u_{2,3(j-1)}\right) \middle| 2 \leq j \leq 6\right\} \cup \\ & \left\{\left(u_{2,j}, u_{3,j+1}\right) \middle| j = 2,4\right\} \cup \left\{\left(u_{2,j}, u_{3,j+3}\right) \middle| j = 5,7\right\} \cup \\ & \left\{\left(u_{2,j}, u_{3,j+5}\right) \middle| j = 8,10\right\} \cup \left\{\left(u_{2,j}, u_{3,j+7}\right) \middle| j = 11,13\right\} \cup \\ & \left\{\left(u_{2,j}, u_{3,j+9}\right) \middle| j = 14,16\right\} \cup \left\{\left(u_{2,j}, u_{3,j+11}\right) \middle| j = 17,19\right\} \cup \\ & \left\{\left(u_{3,j}, u_{4,j+1}\right) \middle| j = 2,4,6\right\} \cup \left\{\left(u_{3,j}, u_{4,j+3}\right) \middle| j = 7,9,11\right\} \cup \\ & \left\{\left(u_{3,j}, u_{4,j+5}\right) \middle| j = 12,14,16\right\} \cup \left\{\left(u_{3,j}, u_{4,j+7}\right) \middle| j = 17,19,21\right\} \cup \\ & \left\{\left(u_{3,j}, u_{4,j+9}\right) \middle| j = 22,24,26\right\} \cup \left\{\left(u_{3,j}, u_{4,j+11}\right) \middle| j = 27,29,31\right\} \cup \\ & \left\{\left(u'_{1,j}, u'_{1,j+1}\right) \middle| 1 \leq j \leq 6\right\} \cup \left\{\left(u'_{2,j}, u'_{2,j+1}\right) \middle| 1 \leq j \leq 18\right\} \cup \\ & \left\{\left(u'_{3,j}, u'_{3,j+1}\right) \middle| 1 \leq j \leq 30\right\} \cup \left\{\left(u'_{4,j}, u'_{4,j+1}\right) \middle| 1 \leq j \leq 42\right\} \cup \\ & \left\{\left(u'_{1,j}, u'_{2,3(j-1)}\right) \middle| 2 \leq j \leq 6\right\} \cup \\ & \left\{\left(u'_{2,j}, u'_{3,j+1}\right) \middle| j = 2,4\right\} \cup \left\{\left(u'_{2,j}, u'_{3,j+3}\right) \middle| j = 5,7\right\} \cup \\ & \left\{\left(u'_{2,j}, u'_{3,j+5}\right) \middle| j = 8,10\right\} \cup \left\{\left(u'_{2,j}, u'_{3,j+7}\right) \middle| j = 11,13\right\} \cup \\ & \left\{\left(u'_{2,j}, u'_{3,j+9}\right) \middle| j = 14,16\right\} \cup \left\{\left(u'_{2,j}, u'_{3,j+11}\right) \middle| j = 17,19\right\} \cup \\ & \left\{\left(u'_{3,j}, u'_{4,j+1}\right) \middle| j = 2,4,6\right\} \cup \left\{\left(u'_{3,j}, u'_{4,j+3}\right) \middle| j = 7,9,11\right\} \cup \\ & \left\{\left(u'_{3,j}, u'_{4,j+5}\right) \middle| j = 12,14,16\right\} \cup \left\{\left(u'_{3,j}, u'_{4,j+7}\right) \middle| j = 17,19,21\right\} \cup \\ & \left\{\left(u'_{3,j}, u'_{4,j+9}\right) \middle| j = 22,24,26\right\} \cup \left\{\left(u'_{3,j}, u'_{4,j+11}\right) \middle| j = 27,29,31\right\} \cup \\ & \left\{\left(u_{1,j}, u'_{1,j}\right) \mid 1 \leq j \leq 6\right\} \cup \left\{\left(u_{2,j}, u'_{2,j}\right) \mid 1 \leq j \leq 18\right\} \cup \\ & \left\{\left(u_{3,j}, u'_{3,j}\right) \mid 1 \leq j \leq 30\right\} \cup \left\{\left(u_{4,j}, u'_{4,j}\right) \mid 1 \leq j \leq 42\right\} \end{aligned}$$

$with\ 7\ mod\ 6 = 19\ mod\ 6 = 31\ mod\ 6 = 43\ mod\ 6 = 1.$

Refer to Theorem 2.1, which is stated that $dim\ (HC(n)) = 3$, with $HC(n) \subseteq K_2 \times HC(n)$ and Theorem 2.2, which is stated that $dim(G) \leq dim(G \times K_2) \leq dim(G) + 1$, then its largest lower bound is $dim\ (K_2 \times HC(4)) \geq 3$. In addition, it will be shown that its smallest upper bound is $dim\ (K_2 \times HC(4)) \leq 3$. Choose the subset $V(K_2 \times HC(4))$, write $W = \left\{u_{4,1}, u'_{4,9}, u_{4,15}\right\}$ so that we obtained the representations for every vertex on graph $K_2 \times HC(4)$ toward W given as follows.

j	$r\left(u_{i,j} \middle\| W\right)$	$r\left(u'_{i,j} \middle\| W\right)$
1 2 3	$r\left(u_{1,j} \middle\| W\right) = [2n-3+j, 2n+2-j, 2n+1-j]$,	$r\left(u'_{1,j} \middle\| W\right) = [2n-2+j, 2n+1-j, 2n+2-j]$,
4 5	$r\left(u_{1,j} \middle\| W\right) = [2n+5-j, 2n-4+j, 2n-5+j]$,	$r\left(u'_{1,j} \middle\| W\right) = [2n+6-j, 2n-5+j, 2n-4+j]$,

$r(u_{1,j}\|W) = [2n-5+j, 2n+8-j, 2n-5+j]$,	$r(u'_{1,j}\|W) = [2n-4+j, 2n+7-j, 2n-4+j]$,
$r(u_{2,j}\|W) = [j+3, 2n+2-j, 4n-3-j]$,	$r(u'_{2,j}\|W) = [j+4, 2n+1-j, 4n-2-j]$,
$r(u_{2,j}\|W) = [j+3, 2n-8+j, 4n-17+j]$,	$r(u'_{2,j}\|W) = [j+4, 2n-9+j, 4n-16+j]$,
$r(u_{2,j}\|W) = [19-j, 2n-8+j, 4n-19+j]$,	$r(u'_{2,j}\|W) = [20-j, 2n-9+j, 4n-18+j]$,
$r(u_{2,j}\|W) = [17-j, 2n+12-j, 4n-19+j]$,	$r(u'_{2,j}\|W) = [18-j, 2n+11-j, 4n-18+j]$,
$r(u_{2,j}\|W) = [j-3, 2n-10+j, 4n-19+j]$,	$r(u'_{2,j}\|W) = [j-2, 2n-11+j, 4n-18+j]$,
$r(u_{2,j}\|W) = [23-j, 2n-18+j, 4n+9-j]$,	$r(u'_{2,j}\|W) = [24-j, 2n-17+j, 4n+10-j]$,
$r(u_{2,j}\|W) = [23-j, 2n+20-j, 4n+11-j]$,	$r(u'_{2,j}\|W) = [24-j, 2n-19+j, 4n+12-j]$,
$r(u_{n-1,j}\|W) = [j+1, 2n+2-j, 4n-3-j]$,	$r(u'_{n-1,j}\|W) = [j+2, 2n+1-j, 4n-2-j]$,
$r(u_{n-1,j}\|W) = [j+1, 2n-12+j, 4n-3-j]$,	$r(u'_{n-1,j}\|W) = [j+2, 2n-13+j, 4n-2-j]$,
$r(u_{n-1,j}\|W) = [25-j, 2n-12+j, 4n-25+j]$,	$r(u'_{n-1,j}\|W) = [26-j, 2n-13+j, 4n-24+j]$,
$r(u_{n-1,j}\|W) = [27-j, 2n-12+j, 4n-25+j]$,	$r(u'_{n-1,j}\|W) = [28-j, 2n-13+j, 4n-24+j]$,
$r(u_{n-1,j}\|W) = [29-j, 2n-12+j, 4n-25+j]$,	$r(u'_{n-1,j}\|W) = [30-j, 2n-13+j, 4n-24+j]$,
$r(u_{n-1,j}\|W) = [31-j, 2n-14+j, 4n-25+j]$,	$r(u'_{n-1,j}\|W) = [32-j, 2n-15+j, 4n-24+j]$,
$r(u_{n-1,j}\|W) = [j-9, 2n-16+j, 4n-25+j]$,	$r(u'_{n-1,j}\|W) = [j-8, 2n-17+j, 4n-24+j]$,
$r(u_{n-1,j}\|W) = [33-j, 2n+28-j, 4n+19-j]$,	$r(u'_{n-1,j}\|W) = [34-j, 2n+27-j, 4n+20-j]$,
$r(u_{n-1,j}\|W) = [33-j, 2n+30-j, 4n+21-j]$,	$r(u'_{n-1,j}\|W) = [34-j, 2n+29-j, 4n+22-j]$,
$r(u_{n-1,j}\|W) = [33-j, 2n+32-j, 4n+23-j]$,	$r(u'_{n-1,j}\|W) = [34-j, 2n+31-j, 4n+24-j]$,
$r(u_{n-1,j}\|W) = [33-j, 2n+32-j, 4n-35+j]$,	$r(u'_{n-1,j}\|W) = [34-j, 2n+31-j, 4n-34+j]$,
$r(u_{n,j}\|W) = [j-1, 2n+2-j, 4n-1-j]$,	$r(u'_{n,j}\|W) = [j, 2n+1-j, 4n-j]$,
$r(u_{n,j}\|W) = [j-1, 2n-16+j, 4n-1-j]$,	$r(u'_{n,j}\|W) = [j, 2n-17+j, 4n-j]$,

17	$r(u_{n,j}\|W) = [31-j, 2n-16+j, 4n-31+j]$,	$r(u'_{n,j}\|W) = [32-j, 2n-17+j, 4n-30+j]$,
18 19	$r(u_{n,j}\|W) = [33-j, 2n-16+j, 4n-31+j]$,	$r(u'_{n,j}\|W) = [34-j, 2n-17+j, 4n-30+j]$,
20 21	$r(u_{n,j}\|W) = [35-j, 2n-16+j, 4n-31+j]$,	$r(u'_{n,j}\|W) = [36-j, 2n-17+j, 4n-30+j]$,
22 23	$r(u_{n,j}\|W) = [37-j, 2n-16+j, 4n-31+j]$,	$r(u'_{n,j}\|W) = [38-j, 2n-17+j, 4n-30+j]$,
24 25	$r(u_{n,j}\|W) = [j-11, 2n+32-j, 4n-31+j]$,	$r(u'_{n,j}\|W) = [j-10, 2n+31-j, 4n-30+j]$,
26 27	$r(u_{n,j}\|W) = [j-13, 2n-20+j, 4n-31+j]$,	$r(u'_{n,j}\|W) = [j-12, 2n-21+j, 4n-30+j]$,
28 29	$r(u_{n,j}\|W) = [j-15, 2n-22+j, 4n-31+j]$,	$r(u'_{n,j}\|W) = [j-14, 2n-23+j, 4n-30+j]$,
30 31	$r(u_{n,j}\|W) = [43-j, 2n+38-j, 4n+29-j]$,	$r(u'_{n,j}\|W) = [44-j, 2n+37-j, 4n+30-j]$,
32 33	$r(u_{n,j}\|W) = [43-j, 2n+40-j, 4n+31-j]$,	$r(u'_{n,j}\|W) = [44-j, 2n+39-j, 4n+32-j]$,
34 35	$r(u_{n,j}\|W) = [43-j, 2n+42-j, 4n+33-j]$,	$r(u'_{n,j}\|W) = [44-j, 2n+41-j, 4n+34-j]$,
36 37 38	$r(u_{n,j}\|W) = [43-j, 2n+44-j, 4n+35-j]$,	$r(u'_{n,j}\|W) = [44-j, 2n+43-j, 4n+36-j]$,
39 40	$r(u_{n,j}\|W) = [43-j, 2n+44-j, 4n+37-j]$,	$r(u'_{n,j}\|W) = [44-j, 2n+43-j, 4n+38-j]$,
41 42	$r(u_{n,j}\|W) = [43-j, 2n+44-j, 4n+39-j]$,	$r(u'_{n,j}\|W) = [44-j, 2n+43-j, 4n+40-j]$.

From the vertex representation above, we obtained that W is the resolving set of a graph $K_2 \times HC(4)$ since $r(v_n|W) \neq r(v_m|W)$ for $\forall n, m \in (K_2 \times HC(4))$ and $n \neq m$. So that, the smallest upper bound of graph $K_2 \times HC(4)$ is $dim\,(K_2 \times HC(4)) \leq 3$. Through this analysis, we obtained the largest lower bound and the smallest upper bound on the metric dimension of graph $K_2 \times HC(4)$ is $3 \leq dim(K_2 \times HC(4)) \leq 3$. Thus, $dim\,(K_2 \times HC(4)) = 3$. ■

Based on Proposition 2.1 and 2.2 above, we formulated the metric dimension of graph $K_2 \times HC(n)$ for $n \in Z^+$ with make a theorem with the following proof.

Theorem 3.1 *Given graph $K_2 \times HC(n)$ for $n \in Z^+$, then $dim\,(K_2 \times HC(n)) = 3$.*
Proof:
We have known before that $3 \leq dim\,(K_2 \times HC(n)) \leq 4$. Choose subset $V(K_2 \times HC(n))$, write $W = \{u_{n,1}, u'_{n,2n+1}, u_{n,4n-1}\}$ for $n \geq 2$ so that we obtained the representation for every vertex of graph $K_2 \times HC(n)$ toward W given as follows.

j	$r(u_{i,j}\|W)$	$r(u'_{i,j}\|W)$
1 2 3	$r(u_{1,j}\|W) = [2n-3+j, 2n+2-j, 2n+1-j]$,	$r(u'_{1,j}\|W) = [2n-2+j, 2n+1-j, 2n+2-j]$,
4 5	$r(u_{1,j}\|W) = [2n+5-j, 2n-4+j, 2n-5+j]$,	$r(u'_{1,j}\|W) = [2n+6-j, 2n-5+j, 2n-4+j]$,
6	$r(u_{1,j}\|W) = [2n-5+j, 2n+8-j, 2n-5+j]$,	$r(u'_{1,j}\|W) = [2n-4+j, 2n+7-j, 2n-4+j]$,
	⋮	
1 2 3	$r(u_{n-1,j}\|W) = [j+1, 2n+2-j, 4n-3-j]$,	$r(u'_{n-1,j}\|W) = [j+2, 2n+1-j, 4n-2-j]$,

4 $\vdots$ $2n-1$		
$\vdots$		
1 2 3 4 5 6 $\vdots$ $2n+1$	$r(u_{n,j}\|W) = [j-1, 2n+2-j, 4n-1-j]$,	$r(u'_{n,j}\|W) = [j, 2n+1-j, 4n-j]$,
$\vdots$		

From the vertex representation above, we obtained that W is the resolving set of a graph $K_2 \times HC(n)$ since $r(v_n|W) \neq r(v_m|W)$ for $\forall n, m \in (K_2 \times HC(n))$ and $n \neq m$. So that we obtained its smallest upper bound is $m\ (K_2 \times HC(4)) \leq 3$. The largest lower bound on metric dimension of graph $K_2 \times HC(n)$ obtained by referring to Theorem 2.1, which is stated that $dim\ (HC(n))\ =\ 3$ and Theorem 2.2, which is stated that $(HC(n)) \subseteq (K_2 \times HC(n))$, with the result that $dim\ (HC(n)) \leq dim\ (K_2 \times HC(n))$, such that $3 \leq dim(G \times K_2) \leq 4$. Based on this explanation before, we obtained the largest lower bound and the smallest upper bound on metric dimension of graph $K_2 \times HC(n)$ is $3 \leq dim\ (K_2 \times HC(n)) \leq 3$. Thus, the metric dimension of graph $K_2 \times HC(n)$ is $dim\ (K_2 \times HC(n)) = 3$ for $n \in Z^+$. ■

3. CONCLUSION

Graph $K_2 \times HC(n)$ is the graph cross operation product of second order complete graph (K_2) with honeycomb network $(HC(n))$. The largest lower bound of the metric dimension for graph $K_2 \times HC(n)$ is $dim\left(K_2 \times HC(n)\right) \geq 3$ and the smallest upper bound of the metric dimension for graph $K_2 \times HC(n)$ is $m\ (K_2 \times HC(n)) \leq 3$. We conclude that the general form of the metric dimension of graph $K_2 \times HC(n)$ using a direct proofing, proven and obtained the exact value, which is $dim\ (K_2 \times HC(n))\ =\ 3$ for $n \in Z^+$.

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