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# Spectral Characteristics of the Anti-adjacency Matrix of Kite Graph

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**Abstract**

Let $G = (V, E)$ be a connected graph, where $V$ is the set of vertices and $E$ is the set of edges of $G$. The kite graph, denoted by $Kite_{n,m}$, is a graph obtained by appending a complete graph $K_n$ to a pendant vertex of path $P_m$. This research investigates the spectrum of anti-adjacency matrix of kite graph. The anti-adjacency matrix of a graph $G$ of order $n$ is a square matrix with order $n$ where the entries of the matrix represent the nonadjacency of the vertices.

**Keywords:** Anti-adjacency matrix, spectrum, characteristic polynomial, complete graph, path.

## 1. INTRODUCTION AND PRELIMINARIES

A graph $G$ is defined as an ordered pair $G = (V, E)$, where $V$ represent the set of vertices and $E$ represents the set of edges that connect pairs of vertices. Specifically, a graph $G$ can be represented by several types of matrices, including the adjacency matrix, incidence matrix, and anti-adjacency matrix. The adjacency matrix of graph $G$, denoted by $A(G)$, is an $n \times n$ matrix where $n$ represents the number of vertices in $G$ and the matrix rows and columns correspond to the vertices of $G$. The entry $a_{ij}$ is 1 if there is an edge connecting $i$ and $j$, and 0 if otherwise. The anti-adjacency matrix of $G$, denoted by $B(G)$, can be seen as the opposite of the adjacency matrix with the entry $b_{ij}$ is 1 if there is no edge connecting $i$ and $j$, and 0 if otherwise. Formally, $B(G)$ can be expressed as $B(G) = J - A(G)$, where $J$ is the matrix with all entries are 1 [3].

The study of various graph matrices offers insights into the structural properties of graphs. One interesting class of graphs for spectral analysis is the kite graph. The kite graph, denoted as $Kite_{n,m}$, is a specific type of graph formed by combining a complete graph of order $n$ with a path of order $m$. It serves as an interesting structure for analyzing the spectral properties of matrices, particularly in exploring how adding paths to complete graphs affects their spectral characteristics. The spectral characteristics that we analyze such as the characteristic polynomial, spectrum, and the determinant. Let $\lambda$ be an eigenvalue of $B(G)$. The spectrum of graph $G$ is defined as the set of eigenvalues of

$B(G)$ with their multiplicities. If the distict eigenvalues of $B(G)$ are $\lambda_1, \lambda_2, \dots, \lambda_n$ with multiplicities $m(\lambda_1), m(\lambda_2), \dots, m(\lambda_n)$, the spectrum can be represented as $Spec\, B(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ m(\lambda_1) & m(\lambda_2) & \dots & m(\lambda_n) \end{pmatrix}$[4]. The spectrum come out as the solution of the characteristic polynomial of $B(G)$, expressed as $p(\lambda) = \det(B(G) - \lambda I) = 0$ [9].

Previous research on the spectral properties of graph matrices has provided valuable insights into their structural and mathematical characteristics. Research conducted by Irawan & Sugeng has explored the properties of the antiadjacency matrix of a graph join, such as its determinant and characteristic polynomial [7]. Putra analyzed the characteristic polynomial and spectrum of the antiadjacency matrix for the corona product between the complete graph $K_m$ and $K_1$ as well as hyper-octahedral graphs $H_m$ and $K_1$ [12]. Yin et al. investigated the spectral characterization of two classes of unicycle graphs [14]. Other result for antiadjacency matrix mostly for directed graph. Anzana et al. investigated the characteristic polynomial of the antiadjacency matrix for directed cyclic friendship graphs [2]. Prayitno et al. explored the properties of the antiadjacency matrix of directed cyclic sun graphs [11]. Research by Aji et al. has focused on the characteristic polynomial and eigenvalue of the antiadjacency matrix of directed unicyclic flower vase graphs [1]. Hasyyati et al. investigated the characteristic polynomial and eigenvalues of the antiadjacency matrix of directed unicyclic corona graphs [6].

For kite graph, Das & Liu explored the spectral properties of adjacency matrix of kite graphs, establishing that these graphs can be uniquely determined by their spectra [5]. Sorgun and Topcu examined the characteristic polynomial of adjacency matrix of the $Kite_{n,m}$ graphs [13].

In this study, the focus is on the spectral properties of the antiadjacency matrix of undirected $Kite_{n,m}$ graph, specifically for $m = 1$, $m = 2$, and $m = 3$. In particular, the $Kite_{n,m}$ with $m = 1$ is commonly known as the "short kite". The study aims to explore key spectral characteristics, including the characteristic polynomial, spectrum, and determinant, through the analysis of the antiadjacency matrix of these graphs.

The following is a representation of the $Kite_{n,m}$ graph with $\{v_1, \dots, v_n\}$ representing the vertex set of $K_n$ and $\{u_1, \dots, u_m\}$ representing the vertex set of $P_m$. Figure 1 shows the $Kite_{n,m}$ graph.

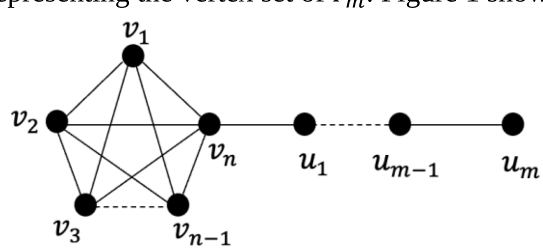


**Figure 1.** The $Kite_{n,m}$ graph

The general form of antiadjacency matrix for the $Kite_{n,m}$ graph is provided below.

$$B(Kite_{n,m}) = \begin{array}{c} \\ v_1 \\ \\ \vdots \\ \\ v_n \\ u_1 \\ \\ \vdots \\ \\ u_m \end{array} \begin{array}{c} \begin{array}{ccccc|ccccc} v_1 & & \cdots & & v_n & u_1 & & \cdots & & u_m \end{array} \\ \left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & 1 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & \cdots & \cdots & \cdots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 1 & \cdots & \cdots & 1 \\ \hline 1 & \cdots & \cdots & 1 & 0 & 1 & 0 & 1 & \cdots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \vdots & 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 & 1 \end{array}\right] \end{array}$$

From the general form of the antiadjacency matrix of the $Kite_{n,m}$ graph, it can be expressed in a block matrix as follows.

$$B(Kite_{n,m}) = \begin{bmatrix} I_{n\times n} & \alpha_{n\times m} \\ \beta_{m\times n} & \gamma_{m\times m} \end{bmatrix}$$

Here, $I$ denotes the identity matrix of size $n \times n$, $\alpha$ is a block matrix of size $n \times m$ with the entry $\alpha_{n1} = 0$ and all other entries are 1, $\beta$ is a block matrix of size $m \times n$ with the entry $\beta_{1n} = 0$ and all other entries are 1, and $\gamma$ is a block matrix of size $m \times m$ with the entries on the superdiagonal and subdiagonal are 0 and all other entries are 1.

To analyze the spectral characteristics of $Kite_{n,1}$, $Kite_{n,2}$, and $Kite_{n,3}$ graphs, several key mathematical concepts and theorems are employed:

**Theorem 1.1**. [10] *If $X$ is an $n \times n$ matrix, then $det\, X = \prod \lambda_i$, where $\lambda_i$ are the eigenvalues of $X$.*

**Theorem 1.2.** [9] *If $X$ is an $n \times n$ matrix and $X = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, then the determinant of $X$ is given by*

$$det\, X = det \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{cases} det(P).\,det(S - RP^{-1}Q) & if\ P\ is\ invertible \\ det(S).\,det(P - QS^{-1}R) & if\ S\ is\ invertible \end{cases}$$

**Vieti's Formulae.** [8] *Let $\mu_1, \mu_2, \dots, \mu_n \in \mathbb{C}$ denote the $n$ roots of the polynomial*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, a_0, a_1, \dots, a_n \in \mathbb{C}, a_0 \neq 0$$

*Then $\mu_1 . \mu_2 . \dots . \mu_n = (-1)^n \left(\frac{a_n}{a_0}\right)$.*

## 2. MAIN RESULTS

The following presents the results obtained for the spectral characteristic of the $Kite_{n,1}$, $Kite_{n,2}$, and $Kite_{n,3}$ graphs.

**Theorem 2.1.** *Let $B$ be the antiadjacency matrix of the graph $Kite_{n,1}$. The spectrum of $B$ consists of $\lambda = 1$ with multiplicity $n - 1$ and $\lambda = 1 \pm \sqrt{n-1}$ with multiplicity 1.*

**Proof.** The form of the antiadjacency matrix of the graph $Kite_{n,1}$ is represented as

$$B(Kite_{n,1}) = \begin{bmatrix} I_{n\times n} & \alpha_{n\times 1} \\ \beta_{1\times n} & \gamma_{1\times 1} \end{bmatrix} = \begin{bmatrix} I_{n\times n} & \alpha_{n\times 1} \\ \beta_{1\times n} & 1 \end{bmatrix}$$

where $\alpha$ is a column vector of size $n \times 1$ and $\beta$ is a row vector of size $1 \times n$. Here, $\gamma$ is simply 1 because $m = 1$. Thus, the characteristic polynomial of the antiadjacency matrix of the graph $Kite_{n,1}$ is

$$p_{B(Kite_{n,1})}(\lambda) = \det\big(B(Kite_{n,1}) - \lambda I\big) = \det \begin{bmatrix} (I - \lambda I)_{n\times n} & \alpha_{n\times 1} \\ \beta_{1\times n} & (1-\lambda)_{1\times 1} \end{bmatrix}$$

Since the block matrix $(I - \lambda I)_{n\times n}$ is invertible, we can apply Theorem 1.2:

$$\begin{aligned} \det B\big(Kite_{n,1}\big) - \lambda I &= \det(I - \lambda I)_{n\times n} \times \det([1-\lambda] - \beta_{1\times n}(I - \lambda I_{n\times n})^{-1}\alpha_{n\times 1}) \\ &= (1-\lambda)^n \times \det\left([1-\lambda] - \beta_{1\times n}\left(\tfrac{1}{1-\lambda}\right) I_{n\times n}\alpha_{n\times 1}\right) \\ &= (1-\lambda)^n \times \det\left([1-\lambda] - \left(\tfrac{1}{1-\lambda}\right)\beta_{1\times n}\alpha_{n\times 1}\right) \\ &= (1-\lambda)^n \times \det\left([1-\lambda] - \left(\tfrac{1}{1-\lambda}\right)[n-1]\right) \\ &= (1-\lambda)^n \times \left(\frac{n - \lambda^2 + 2\lambda - 2}{\lambda - 1}\right) \\ &= (1-\lambda)^{n-1} \times \big(\lambda^2 - 2\lambda + (2-n)\big) \end{aligned}$$

Therefore, the characteristic polynomial of the graph $Kite_{n,1}$ is given by:

$$p_{B(Kite_{n,1})}(\lambda) = (1-\lambda)^{n-1}\big(\lambda^2 - 2\lambda + (2-n)\big)$$

Based on the characteristic polynomial above, the spectrum of the antiadjacency matrix of the graph $Kite_{n,1}$ can be determined.
For the first factor: $(1-\lambda)^{n-1}$
The eigenvalue associated with this factor is $\lambda = 1$. This because when $\lambda = 1$, $(1-\lambda)^{n-1}$ becomes zero. So, $\lambda = 1$ is an eigenvalue with multiplicity $n-1$.
For the second factor: $\left(\lambda^2 - 2\lambda + (2-n)\right)$
Using quadratic formula, so we obtain the eigenvalues from this factor are $\lambda_1 = 1-\sqrt{n-1}$ and $\lambda_2 = 1+\sqrt{n-1}$. Combining the results from both factors, the spectrum of the antiadjacency matrix of the $Kite_{n,1}$ graph are

$$Spec\ B\left(Kite_{n,1}\right) = \begin{pmatrix} 1 & 1 \pm \sqrt{n-1} \\ n-1 & 1 \end{pmatrix} \qquad \blacksquare$$

**Corollary 2.2.** *Let $B$ be the antiadjacency of the graph $Kite_{n,1}$. The determinant of $B$ is given by*

$$det\left(B\left(Kite_{n,1}\right)\right) = 2-n$$

**Proof.** From Theorem 2.1, the spectrum of the antiadjacency matrix of the graph $Kite_{n,1}$ are 1 with multiplicity $n-1$, and $1 \pm \sqrt{n-1}$ with multiplicity 1. According to Theorem 1.1, the determinant of the matrix is the product of all its eigenvalues. Therefore

$$\begin{aligned} \det B\left(Kite_{n,1}\right) &= \prod \lambda_i \\ &= (1)^{n-1}\left(1+\sqrt{n-1}\right)\left(1-\sqrt{n-1}\right) \\ &= 1\left(1-(n-1)\right) \\ &= 2-n \end{aligned} \qquad \blacksquare$$

**Theorem 2.3.** *Let $B$ be the antiadjacency matrix of $Kite_{n,2}$. The spectrum of $B$ consists of $\lambda = 1$ with multiplicity $n-2$, $\lambda = 1 \pm \frac{\sqrt{2}\sqrt{2n-\sqrt{4n^2-8n+5}-1}}{2}$ with multiplicity 1, and $\lambda = 1 \pm \frac{\sqrt{2}\sqrt{2n+\sqrt{4n^2-8n+5}-1}}{2}$ with multiplicity 1.*

**Proof.** The form of the antiadjacency matrix of the graph $Kite_{n,2}$ is represented as

$$B\left(Kite_{n,2}\right) = \begin{bmatrix} I_{n\times n} & \alpha_{n\times 2} \\ \beta_{2\times n} & \gamma_{2\times 2} \end{bmatrix} = \begin{bmatrix} I_{n\times n} & \alpha_{n\times 2} \\ \beta_{2\times n} & I_{2\times 2} \end{bmatrix}$$

where $\alpha$ is an $n \times 2$ matrix and $\beta$ is a $2 \times n$ matrix. Here, $\gamma$ is $2 \times 2$ identity matrix. Thus, the characteristic polynomial of the antiadjacency matrix of the graph $Kite_{n,2}$ is

$$p_{B(Kite_{n,2})}(\lambda) = \det\left(B\left(Kite_{n,2}\right) - \lambda I\right) = \det \begin{bmatrix} (I-\lambda I)_{n\times n} & \alpha_{n\times 2} \\ \beta_{2\times n} & (I-\lambda I)_{2\times 2} \end{bmatrix}$$

Since the block matrix $(I-\lambda I)_{n\times n}$ is invertible, we can apply Theorem 1.2:

$$\begin{aligned} \det B\left(Kite_{n,2}\right) - \lambda I &= \det(I-\lambda I)_{n\times n} \times \det((I-\lambda I)_{2\times 2} - \beta_{2\times n}(I-\lambda I_{n\times n})^{-1}\alpha_{n\times 2}) \\ &= (1-\lambda)^n \times \det\left((I-\lambda I)_{2\times 2} - \beta_{2\times n}\left(\tfrac{1}{1-\lambda}\right) I_{n\times n}\, \alpha_{n\times 2}\right) \\ &= (1-\lambda)^n \times \det\left((I-\lambda I)_{2\times 2} - \left(\tfrac{1}{1-\lambda}\right)\beta_{2\times n}\, \alpha_{n\times 2}\right) \\ &= (1-\lambda)^n \times \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} - \left(\tfrac{1}{1-\lambda}\right)\begin{bmatrix} n-1 & n-1 \\ n-1 & n \end{bmatrix}\right) \\ &= (1-\lambda)^n \times \det \begin{bmatrix} \dfrac{n-\lambda^2+2\lambda-2}{\lambda-1} & \dfrac{n-1}{\lambda-1} \\ \dfrac{n-1}{\lambda-1} & \dfrac{n-\lambda^2+2\lambda-1}{\lambda-1} \end{bmatrix} \end{aligned}$$

$$= (1-\lambda)^n \times \left(\frac{-n+\lambda^4-4\lambda^3-2n\lambda^2+7\lambda^2+4n\lambda-6\lambda+1}{\lambda^2-2\lambda+1}\right)$$
$$= \frac{(1-\lambda)^n(\lambda^4-4\lambda^3+(7-2n)\lambda^2+(4n-6)\lambda+(1-n))}{(1-\lambda)^2}$$
$$= (1-\lambda)^{n-2}(\lambda^4-4\lambda^3+(7-2n)\lambda^2+(4n-6)\lambda+(1-n))$$

Therefore, the characteristic polynomial of the graph $Kite_{n,2}$ is given by:

$$p_{B(Kite_{n,2})}(\lambda) = (1-\lambda)^{n-2}(\lambda^4-4\lambda^3+(7-2n)\lambda^2+(4n-6)\lambda+(1-n))$$

Based on the characteristic polynomial above, the spectrum of the antiadjacency matrix of the graph $Kite_{n,2}$ can be determined.
For the first factor: $(1-\lambda)^{n-2}$
The eigenvalue associated with this factor is $\lambda = 1$. Because, when $\lambda = 1$, $(1-\lambda)^{n-2}$ becomes zero. So, $\lambda = 1$ is an eigenvalue with multiplicity $n-2$.
For the second factor: $(\lambda^4-4\lambda^3+(7-2n)\lambda^2+(4n-6)\lambda+(1-n))$
To find the eigenvalues corresponding to this factor, factorize the polynomial. Due to the complexity of the polynomial, the factorization was carried out using computational tools. Using Wolfram Mathematica, the quadratic equations were solved. The following eigenvalues were obtained from this factor.

$$\lambda_{1,2} = 1 \pm \frac{\sqrt{2}\sqrt{2n-\sqrt{4n^2-8n+5}-1}}{2}$$
$$\lambda_{3,4} = 1 \pm \frac{\sqrt{2}\sqrt{2n+\sqrt{4n^2-8n+5}-1}}{2}$$

Therefore, the spectrum of antiadjacency matrix of the graph $Kite_{n,2}$ are

$$Spec\ B(Kite_{n,2}) = \begin{pmatrix} 1 & 1 \pm \frac{\sqrt{2}\sqrt{2n-\sqrt{4n^2-8n+5}-1}}{2} & 1 \pm \frac{\sqrt{2}\sqrt{2n+\sqrt{4n^2-8n+5}-1}}{2} \\ n-2 & 1 & 1 \end{pmatrix} \quad \blacksquare$$

**Corollary 2.4.** *Let $B$ be the antiadjacency of the graph $Kite_{n,2}$. The determinant of $B$ is given by*

$$det\left(B(Kite_{n,2})\right) = 1-n$$

**Proof.** From Theorem 2.3, the characteristic polynomial of $B(Kite_{n,2})$ is given by

$$p_{B(Kite_{n,2})}(\lambda) = (1-\lambda)^{n-2}(\lambda^4-4\lambda^3+(7-2n)\lambda^2+(4n-6)\lambda+(1-n))$$

To find the determinant, we need the product of all its eigenvalues. This involves analyzing the roots of the second factor

$$\lambda^4-4\lambda^3+(7-2n)\lambda^2+(4n-6)\lambda+(1-n)$$

Since $a_0 = 1$ and $a_4 = 1-n$, by Vieti's formulas, the product of all the roots from this factor is given by

$$\lambda_1\lambda_2\lambda_3\lambda_4 = (-1)^4\frac{1-n}{1} = 1-n$$

The eigenvalue $\lambda = 1$ appears with multiplicity $n-2$ from the first factor and the product of the roots from second factor is $1-n$. Hence, the determinant of the antiadjacency matrix of the graph $Kite_{n,2}$ is

$$\det B(Kite_{n,2}) = \prod \lambda_i$$
$$= (1)^{n-2}(1-n)$$
$$= 1-n \quad \blacksquare$$

**Theorem 2.5.** *Let $B$ be the antiadjacency matrix of $Kite_{n,3}$. The spectrum of $B$ consists of $\lambda = 1$ with multiplicity $n-1$, $\lambda = 1 - \frac{1}{2}\left(\sqrt{2n+P+Q} \pm \sqrt{4n-P-Q-\frac{4(-1+n)}{\sqrt{2n+P+Q}}}\right)$ with multiplicity 1, and $\lambda = 1 + \frac{1}{2}\left(\sqrt{2n+P+Q} \pm \sqrt{4n-P-Q+\frac{4(-1+n)}{\sqrt{2n+P+Q}}}\right)$ with multiplicity 1, where $P$ and $Q$ are defined as*

$$P = \frac{\left((-108n+126n^2-9n^3) + 2\left(9+\sqrt{15(31-180n+414n^2-459n^3+243n^4-54n^5)}\right)\right)^{\frac{1}{3}}}{3^{\frac{2}{3}}}$$

$$Q = \frac{(-8+12n+3n^2)}{\left((-324n+378n^2-27n^3)+6\left(9+\sqrt{15(31-180n+414n^2-459n^3+243n^4-54n^5)}\right)\right)^{\frac{1}{3}}}$$

**Proof.** The form of the antiadjacency matrix of the graph $Kite_{n,3}$ is represented as

$$B(Kite_{n,3}) = \begin{bmatrix} I_{n\times n} & \alpha_{n\times 3} \\ \beta_{3\times n} & \gamma_{3\times 3} \end{bmatrix}$$

where $\alpha$ is an $n \times 3$ matrix and $\beta$ is a $3 \times n$ matrix. Here, $\gamma$ is a block matrix of size $3 \times 3$ with the entries on the superdiagonal and subdiagonal equal to 0 and all other entries equal to 1. Thus, the characteristic polynomial of the antiadjacency matrix of the graph $Kite_{n,3}$ is

$$p_{B(Kite_{n,3})}(\lambda) = \det\left(B(Kite_{n,3}) - \lambda I\right) = \det\begin{bmatrix} (I-\lambda I)_{n\times n} & \alpha_{n\times 3} \\ \beta_{3\times n} & (\gamma-\lambda I)_{3\times 3} \end{bmatrix}$$

Since the block matrix $(I-\lambda I)_{n\times n}$ is invertible, we can apply Theorem 1.2:

$$\begin{aligned}
\det B(Kite_{n,3}) - \lambda I &= \det(I-\lambda I)_{n\times n} \times \det((\gamma-\lambda I)_{3\times 3} - \beta_{3\times n}(I-\lambda I)^{-1}\alpha_{n\times 3}) \\
&= (1-\lambda)^n \times \det\left((\gamma-\lambda I)_{3\times 3} - \beta_{3\times n}\left(\frac{1}{1-\lambda}\right) I_{n\times n}\alpha_{n\times 3}\right) \\
&= (1-\lambda)^n \times \det\left((\gamma-\lambda I)_{3\times 3} - \left(\frac{1}{1-\lambda}\right)\beta_{3\times n}\alpha_{n\times 3}\right) \\
&= (1-\lambda)^n \times \det\left(\begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} - \begin{bmatrix} \frac{n-1}{1-\lambda} & \frac{n-1}{1-\lambda} & \frac{n-1}{1-\lambda} \\ \frac{n-1}{1-\lambda} & \frac{n}{1-\lambda} & \frac{n}{1-\lambda} \\ \frac{n-1}{1-\lambda} & \frac{n}{1-\lambda} & \frac{n}{1-\lambda} \end{bmatrix}\right) \\
&= (1-\lambda)^n \times \det\begin{bmatrix} \frac{\lambda^2-2\lambda-n+2}{1-\lambda} & \frac{1-n}{1-\lambda} & \frac{2-\lambda-n}{1-\lambda} \\ \frac{1-n}{1-\lambda} & \frac{\lambda^2-2\lambda-n+1}{1-\lambda} & \frac{-n}{1-\lambda} \\ \frac{2-\lambda-n}{1-\lambda} & \frac{-n}{1-\lambda} & \frac{\lambda^2-2\lambda-n+1}{1-\lambda} \end{bmatrix} \\
&= (1-\lambda)^n \times \left(\left(\frac{\lambda^2-2\lambda-n+2}{1-\lambda}\right)\left(\frac{\lambda^2-2\lambda-n+1}{1-\lambda}\right)\left(\frac{\lambda^2-2\lambda-n+1}{1-\lambda}\right) + \right. \\
&\left(\frac{1-n}{1-\lambda}\right)\left(\frac{-n}{1-\lambda}\right)\left(\frac{2-\lambda-n}{1-\lambda}\right) + \left(\frac{2-\lambda-n}{1-\lambda}\right)\left(\frac{1-n}{1-\lambda}\right)\left(\frac{-n}{1-\lambda}\right) - \\
&\left(\frac{2-\lambda-n}{1-\lambda}\right)\left(\frac{\lambda^2-2\lambda-n+1}{1-\lambda}\right)\left(\frac{2-\lambda-n}{1-\lambda}\right) - \left(\frac{-n}{1-\lambda}\right)\left(\frac{-n}{1-\lambda}\right)\left(\frac{\lambda^2-2\lambda-n+2}{1-\lambda}\right) - \\
&\left.\left(\frac{\lambda^2-2\lambda-n+1}{1-\lambda}\right)\left(\frac{1-n}{1-\lambda}\right)\left(\frac{1-n}{1-\lambda}\right)\right)
\end{aligned}$$

$$= (1-\lambda)^n \left(\frac{\lambda^4-4\lambda^3+6\lambda^2-2\lambda-3\lambda^2 n+4\lambda n+2n-3}{1-\lambda}\right)$$
$$= (1-\lambda)^{n-1}(\lambda^4-4\lambda^3+6\lambda^2-2\lambda-3\lambda^2 n+4\lambda n+2n-3)$$
$$= (1-\lambda)^{n-1}\left(\lambda^4-4\lambda^3+(6-3n)\lambda^2+(4n-2)\lambda+(2n-3)\right)$$

Therefore, the characteristic polynomial of the graph $Kite_{n,3}$ is given by:

$$p_{B(Kite_{n,3})}(\lambda) = (1-\lambda)^{n-1}\left(\lambda^4-4\lambda^3+(6-3n)\lambda^2+(4n-2)\lambda+(2n-3)\right)$$

Based on the characteristic polynomial above, the spectrum of the antiadjacency matrix of the graph $Kite_{n,3}$ can be determined.

For the first factor: $(1-\lambda)^{n-1}$

The eigenvalue associated with this factor is $\lambda = 1$. Because, when $\lambda = 1$, $(1-\lambda)^{n-1}$ becomes zero. So, $\lambda = 1$ is an eigenvalue with multiplicity $n-1$.

For the second factor: $\left(\lambda^4-4\lambda^3+(6-3n)\lambda^2+(4n-2)\lambda+(2n-3)\right)$

Given the complexity of this polynomial, the factorization was carried out using computational tools. Using Wolfram Mathematica, the following results were obtained. Let

$$P = \frac{\left((-108n+126n^2-9n^3\ )+2\left(9+\sqrt{15(31-180n+414n^2-459n^3+243n^4-54n^5)}\right)\right)^{\frac{1}{3}}}{3^{\frac{2}{3}}}$$

and

$$Q = \frac{(-8+12n+3n^2)}{\left((-324n+378n^2-27n^3)+6\left(9+\sqrt{15(31-180n+414n^2-459n^3+243n^4-54n^5)}\right)\right)^{\frac{1}{3}}}$$

Then, the eigenvalues are

$$\lambda_{1,2} = 1-\frac{1}{2}\left(\sqrt{2n+P+Q} \pm \sqrt{4n-P-Q-\frac{4(-1+n)}{\sqrt{2n+P+Q}}}\right)$$
$$\lambda_{3,4} = 1+\frac{1}{2}\left(\sqrt{2n+P+Q} \pm \sqrt{4n-P-Q+\frac{4(-1+n)}{\sqrt{2n+P+Q}}}\right)$$

Thus, the spectrum of the antiadjacency matrix of the graph $Kite_{n,3}$ are

$$Spec\ B(Kite_{n,3}) = \begin{pmatrix} 1 & 1-\frac{1}{2}\left(\sqrt{2n+P+Q} \pm \sqrt{4n-P-Q-\frac{4(-1+n)}{\sqrt{2n+P+Q}}}\right) & 1+\frac{1}{2}\left(\sqrt{2n+P+Q} \pm \sqrt{4n-P-Q+\frac{4(-1+n)}{\sqrt{2n+P+Q}}}\right) \\ n-1 & 1 & 1 \end{pmatrix} \blacksquare$$

**Corollary 2.6.** *Let $B$ be the antiadjacency of the graph $Kite_{n,3}$. The determinant of $B$ is given by*

$$det\left(B(Kite_{n,3})\right) = 2n-3$$

**Proof.** From Theorem 2.5, the characteristic polynomial of $B(Kite_{n,3})$ is given by

$$p_{B(Kite_{n,3})}(\lambda) = (1-\lambda)^{n-1}\left(\lambda^4-4\lambda^3+(6-3n)\lambda^2+(4n-2)\lambda+(2n-3)\right)$$

To find the determinant, we need the product of all its eigenvalues. This involves analyzing the roots of the second factor

$$\lambda^4-4\lambda^3+(6-3n)\lambda^2+(4n-2)\lambda+(2n-3)$$

Since $a_0 = 1$ and $a_4 = 2n-3$, by Vieti's formulas, the product of all the roots from this factor is given by

$$\lambda_1\lambda_2\lambda_3\lambda_4 = (-1)^4 \frac{2n-3}{1} = 2n-3$$

The eigenvalue $\lambda = 1$ appears with multiplicity $n-1$ from the first factor and the product of the roots from second factor is $2n-3$. Hence, the determinant of the antiadjacency matrix of the graph $Kite_{n,3}$ is

$$\begin{aligned}\det B(Kite_{n,3}) &= \prod \lambda_i \\ &= (1)^{n-1}(2n-3) \\ &= 2n-3\end{aligned}$$

∎

## 3. CONCLUSION

According to the result above, we obtain the spectral characteristics of antiadjacency matrix of the graph $Kite_{n,1}$, $Kite_{n,2}$, and $Kite_{n,3}$. Each graph exhibits distinct spectral properties, including unique characteristic polynomials, spectrum, and determinant. For further research, we can investigate the characteristic polynomial, spectrum, and determinant of the graph $Kite_{n,m}$ for arbitary values of $m$. This exploration could provide deeper insight into the spectral properties of general kite graphs and their potential applications.

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## CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this paper.

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