Some New Properties of $g$-Frame in Hilbert $C^*$-Modules

Mohamed Rossafi$^1$, Hatim Labrigui$^2$

Abstract

The theory of frames which appeared in the last half of the century, has been generalized rapidly and various generalizations of frames in Hilbert spaces and Hilbert $C^*$-modules. In this paper, we will give some new properties of modular Riesz basis and modular $g$-Riesz basis that present a generalization of the results established in a Hilbert space.

Keywords: Frame, modular Riesz basis, modular $g$-Riesz basis, $C^*$-algebra, Hilbert $A$-modules.

1. INTRODUCTION

Frame theory has a great revolution for recent years, this theory has several properties applicable in many fields of mathematics and engineering and play a significant role in signal and image processing, which leads to many applications in informatics, medicine and probability. Frame theory has been extended from Hilbert spaces to Hilbert $C^*$-modules and began to be study widely and deeply. The basic idea was to consider module over $C^*$-algebra instead of linear spaces and to allow the inner product to take values in the $C^*$-algebra.

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [4] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [3] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [6]. Frames have been used in signal processing, image processing, data compression and sampling theory.

This theory has been extended by Frank and Larson [5] in 2000 for the elements of $C^*$-algebras and Hilbert $C^*$-modules. Eventually, frames with $C^*$-valued bounds in Hilbert $C^*$-modules have been considered in [1].
The theory of frames has been generalized rapidly and there are various generalizations of frames in Hilbert spaces and Hilbert $C^*$-modules. The notions of modular Riesz basis and modular $g$-Riesz basis has been introcued by Khosravi A and Khosravi B in [10].

The aim of this paper is to extend results of Khosravi A and Farmani M. R [8], given in the Hilbert space to Hilbert $C^*$-module.

Let us recall some definitions and basic properties of a Hilbert $C^*$-module that we need in the rest of the paper.

For a $C^*$-algebra $A$ if $a \in A$ is positive we write $a \geq 0$ and $A^+$ denotes the set of all positive elements of $A$.

\begin{definition}{1.1} [11] Let $A$ be a unital $C^*$-algebra and $H$ be a left $A$-module, such that the linear structures of $A$ and $H$ are compatible. $H$ is a pre-Hilbert $A$-module if $H$ is equipped with an $A$-valued inner product $\langle ., . \rangle_A : H \times H \rightarrow A$, such that is sesquilinear, positive definite and respects the module action. In the other words,

\begin{enumerate}
  \item $\langle x, x \rangle_A \geq 0$, for all $x \in H$, and $\langle x, x \rangle_A = 0$ if and only if $x = 0$.
  \item $\langle ax + y, z \rangle_A = a\langle x, z \rangle_A + \langle y, z \rangle_A$, for all $a \in A$ and $x, y, z \in H$.
  \item $\langle x, y \rangle_A = \langle y, x \rangle_A^*$, for all $x, y \in H$.
\end{enumerate}

For $x \in H$, we define the norm of $x$ by $\|x\| = \|\langle x, x \rangle_A\|^{\frac{1}{2}}_A$. If $H$ is complete with $\| . \|$, it is called a Hilbert $A$-module or a Hilbert $C^*$-modules over $A$.

For every $a$ in $C^*$-algebra $A$, we have $|a|_A = (a^*a)^{\frac{1}{2}}$ and the $A$-valued norm on $H$ is defined by $|x| = \langle x, x \rangle_A^{\frac{1}{2}}$, for all $x \in H$.

Let $H$ and $K$ be two Hilbert $A$-modules, a map $T : H \rightarrow K$ is said to be adjointable if there exists a map $T^* : K \rightarrow H$ such that $\langle Tx, y \rangle_A = \langle x, T^*y \rangle_A$ for all $x \in H$ and $y \in K$.

Throughout this paper, We reserve the notation $End_A^*(H, K)$ for the set of all adjointable operators from $H$ to $K$ and $End_A^*(H, H)$ is abbreviated to $End_A^*(H)$.

For a unital $C^*$-algebra $A$, let $I$ and $J$ be a finite or countable subset of $\mathbb{Z}$ and $\{H_i\}_{i \in I}$ be a sequence of Hilbert $A$-modules. Let $l^2(\{H_i\}_{i \in I})$ be the Hilbert $A$-module defined by

$$l^2(\{H_i\}_{i \in I}) = \left\{ \{x_i\}_{i \in I} : x_i \in H_i, \sum_{i \in I} \langle x_i, x_i \rangle_A \text{ converge in } \| . \| \right\}.$$
Let $l^2(A)$ be the Hilbert $A$-module defined by

\[ l^2(A) = \left\{ \{a_j\}_{j \in J} \subseteq A : \sum_{j \in J} a_j a_j^* \text{ converge in } \| \cdot \| \right\} \]

The following lemmas will be used to prove our main results.

**Lemma 1.2.** [2]. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $A$-modules and $T \in \text{End}_A^*(\mathcal{H}, \mathcal{K})$. The following statements are equivalent:

(i) $T$ is surjective.

(ii) $T^*$ is bounded below with respect to norm, i.e. there is $m > 0$ such that

\[ m\|x\| \leq \|T^*x\| \quad x \in \mathcal{K}. \]

(iii) $T^*$ is bounded below with respect to the inner product, i.e. there is $m' > 0$ such that,

\[ m'\langle x, x \rangle_A \leq \langle T^*x, T^*x \rangle_A \quad x \in \mathcal{K}. \]

**Lemma 1.3.** [11]. Let $\mathcal{H}$ be a Hilbert $A$-module and $T \in \text{End}_A^*(\mathcal{H})$, then we have for all $x \in \mathcal{H}$,

\[ \langle Tx, Tx \rangle_A \leq \|T\|^2 \langle x, x \rangle_A. \]

**Lemma 1.4.** [7] Let $A$ be a $C^*$-algebra. Suppose that $\{a_j\}_{j \in J}$ and $\{b_j\}_{j \in J}$ are two sequences of $A$ such that both $\sum_{j \in J} a_j a_j^*$ and $\sum_{j \in J} b_j b_j^*$ converge in $A$, then,

\[ \sum_{j \in J} (a_j + b_j)(a_j + b_j)^* \leq 2 \sum_{j \in J} (a_j a_j^* + b_j b_j^*) \]

2. G-frame in Hilbert $C^*$-Module

**Definition 2.1.** [5]. Let $\mathcal{H}$ be a Hilbert $A$-module. A family $\{x_i\}_{i \in I}$ of elements of $\mathcal{H}$ is called a frame for $\mathcal{H}$, if there exist two positive constants $A$, $B$, such that for all $x \in \mathcal{H}$,

\[ A\langle x, x \rangle_A \leq \sum_{i \in I} \langle x, x_i \rangle_A \langle x_i, x \rangle_A \leq B\langle x, x \rangle_A. \]

The numbers $A$ and $B$ are called lower and upper bound of the frame, respectively.

If $A = B = \lambda$, the frame is called $\lambda$-tight. If $A = B = 1$, it is called a normalized tight frame or a Parseval frame. If only upper inequality of (2.1) hold, then $\{x_i\}_{i \in I}$ is called a Bessel sequence for $\mathcal{H}$.
Let \( \{x_i\}_{i \in I} \) be a Bessel sequence in a Hilbert \( C^* \)-module \( \mathcal{H} \), we define the analysis operator by

\[ T : \mathcal{H} \rightarrow l^2(\mathcal{A}) \]
\[ x \rightarrow \{\langle x, x_i \rangle_{\mathcal{A}}\}_{i \in I} \]

\( T \) is a bounded linear operator, the adjoin operator called the synthesis operator is defined by

\[ T^* : l^2(\mathcal{A}) \rightarrow \mathcal{H} \]
\[ \{a_i\}_{i \in I} \rightarrow \sum_{i \in I} a_i x_i \]

By composing \( T \) and \( T^* \), the frame operator \( S \) is given by

\[ S : \mathcal{H} \rightarrow \mathcal{H} \]
\[ x \rightarrow Sx = T^* Tx = \sum_{i \in I} \langle x, x_i \rangle_{\mathcal{A}} x_i \]

**Proposition 2.2.** Let \( \{x_i\}_{i \in I} \) be a frame in a Hilbert \( C^* \)-module \( \mathcal{H} \). Then the frame operator \( S \) thus defined is bounded, selfadjoint, positive and invertible.

**Definition 2.3.** [9]. Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module and \( (\mathcal{H}_i)_{i \in I} \) be a sub-modules of \( \mathcal{H} \). We call a sequence \( \{\Lambda_i \in \text{End}^*_\mathcal{A}(\mathcal{H}, \mathcal{H}_i), i \in I\} \) a \( g \)-frame in Hilbert \( \mathcal{A} \)-module \( \mathcal{H} \) with respect to \( \{\mathcal{H}_i\}_{i \in I} \) if there exist two positive constants \( A \) and \( B \), such that for all \( x \in \mathcal{H} \),

\[ A \langle x, x \rangle_{\mathcal{A}} \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \leq B \langle x, x \rangle_{\mathcal{A}}. \tag{2.2} \]

The numbers \( A \) and \( B \) are called lower and upper bound of the \( g \)-frame, respectively.

If \( A = B = \lambda \), the \( g \)-frame is called a \( \lambda \)-tight. If \( A = B = 1 \), it is called a \( g \)-Parseval frame.

Let \( \mathcal{H} \) and \( \mathcal{K} \) be a Hilbert \( \mathcal{A} \)-modules. We recall that \( \mathcal{H} \oplus \mathcal{K} = \{(x, y) : x \in \mathcal{H}, y \in \mathcal{K}\} \) is a Hilbert \( \mathcal{A} \)-module with pointwise operations and inner product

\[ \langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle_{\mathcal{A}} + \langle y_1, y_2 \rangle_{\mathcal{A}} \quad x_1, x_2 \in \mathcal{H}; \ y_1, y_2 \in \mathcal{K}. \]

Let \( \mathcal{U} \) and \( \mathcal{V} \) be two Hilbert \( \mathcal{A} \)-modules. For \( T \in \text{End}^*_\mathcal{A}(\mathcal{H}, \mathcal{U}) \) and \( L \in \text{End}^*_\mathcal{A}(\mathcal{K}, \mathcal{V}) \) we define

\[ T \oplus L \in \text{End}^*_\mathcal{A}(\mathcal{H} \oplus \mathcal{K}, \mathcal{U} \oplus \mathcal{V}) \quad \text{by} \quad (T \oplus L)(x, y) := (Tx, Ly) \quad x \in \mathcal{H}, \ y \in \mathcal{K}. \]
Theorem 2.4. Let \( \{x_i\}_{i \in I} \) be a frame for \( \mathcal{H} \) with bounds \( A, B \) and frame operator \( S_x \). Let \( \{y_j\}_{j \in J} \) be a frame for \( \mathcal{K} \) with bounds \( C, D \) and frame operator \( S_y \). Then \( \{(x_i \oplus y_j)\}_{(i \in I, j \in J)} \) is a frame for \( \mathcal{H} \oplus \mathcal{K} \) with frame operator \( S_{(x \oplus y)} = S_x \oplus S_y \).

Proof. Let \( \{x_i\}_{i \in I} \) and \( \{y_j\}_{j \in J} \) be a frames as they were defined in the last Theorem. Then for all \( x \in \mathcal{H} \) and \( y \in \mathcal{K} \) we have

(2.3) \[ A \langle x, x \rangle_A \leq \sum_{i \in I} (x, x_i)_A \langle x, x \rangle_A \leq B \langle x, x \rangle_A. \]

(2.4) \[ C \langle y, y \rangle_A \leq \sum_{j \in J} (y, y_i)_A \langle y, y \rangle_A \leq D \langle y, y \rangle_A. \]

From (2.3), (2.4) and Lemma (1.4), we have

\[
\min \{A, C\}(\langle (x, y), (x, y) \rangle) = \min \{A, C\}(\langle x, x \rangle_A + \langle y, y \rangle_A) \\
\leq \sum_{(i, j) \in I \times J} |\langle (x, y), (x_i, y_j) \rangle|^2 \\
= \sum_{(i, j) \in I \times J} |\langle x, x_i \rangle_A + \langle y, y_j \rangle_A|^2 \\
\leq 2 \max \{B, D\}(\langle x, x \rangle_A + \langle y, y \rangle_A) \\
= 2 \max \{B, D\}(\langle (x, y), (x, y) \rangle),
\]

which shows that \( \{(x_i \oplus y_j)\}_{(i \in I, j \in J)} \) is a frame for \( \mathcal{H} \oplus \mathcal{K} \).

Moreover, we have for all \( (x \oplus y) \in \mathcal{H} \oplus \mathcal{K} \),

\[
S_{(x \oplus y)}(x, y) = \sum_{i \in I, j \in J} (\langle (x, y), (x_i, y_j) \rangle)(x_i, y_j) \\
= \sum_{i \in I, j \in J} (\langle x, x_i \rangle_A + \langle y, y_i \rangle_A)(x_i, y_j) \\
= S_x(x) \oplus S_y(y) = (S_x \oplus S_y)(x \oplus y).
\]

Then,

\[ S_{(x \oplus y)} = S_x \oplus S_y \]

\( \square \)
3. Modular G-frame in Hilbert C*-Module

Let \( \{\Lambda_i\}_{i \in I} \) be a \( g \)-Bessel sequence for \( \{\text{End}_A^*(\mathcal{H}, \mathcal{H}_i), i \in I\} \), we recall that the analysis operator for \( \{\Lambda_i\}_{i \in I} \) is defined by

\[
T_\Lambda : \mathcal{H} \rightarrow l^2(\{\mathcal{H}_i\}_{i \in I})
\]

\[
x \mapsto T_\Lambda^*x = \{\Lambda_i x\}_{i \in I}
\]

The adjoin of this operator is called the synthesis operator and defined by

\[
T_\Lambda^* : l^2(\{\mathcal{H}_i\}_{i \in I}) \rightarrow \mathcal{H}
\]

\[
\{x_i\}_{i \in I} \mapsto T_\Lambda^*\{x_i\}_{i \in I} = \sum_{i \in I} \Lambda_i^* x_i.
\]

The \( g \)-frame operator \( S_\Lambda \) is defined by

\[
S_\Lambda : \mathcal{H} \rightarrow \mathcal{H}
\]

\[
x \mapsto S_\Lambda x = T_\Lambda^* T_\Lambda x = \sum_{i \in I} \Lambda_i^* \Lambda_i x.
\]

The \( g \)-frame operator \( S_\Lambda \) is a positive and self-adjoin operator. Moreover, if \( \{\Lambda_i\}_{i \in I} \) is a \( g \)-frame, then \( S_\Lambda \) is invertible, for more details see ([9])

**Definition 3.1.** [12] Let \( K \in \text{End}_A^*(\mathcal{H}) \) and \( \Lambda_i \in \text{End}_A^*(\mathcal{H}, \mathcal{H}_i) \) for all \( i \in I \), then \( \{\Lambda_i\}_{i \in I} \) is said to be a \( K \)-\( g \)-frame for \( \mathcal{H} \) with respect to \( \{\mathcal{H}_i\}_{i \in I} \) if there exist two constants \( A, B > 0 \) such that

\[
A \langle K^* x, K^* x \rangle_A \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_A \leq B \langle x, x \rangle_A \quad x \in \mathcal{H}.
\]

**Theorem 3.2.** Let \( K \in \text{End}_A^*(\mathcal{H}) \) and let \( \{\Lambda_i \in \text{End}_A^*(\mathcal{H}, \mathcal{H}_i), i \in I\} \) be a \( g \)-frame in Hilbert \( A \)-module \( \mathcal{H} \) with respect to \( \{\mathcal{H}_i\}_{i \in I} \) with bounds \( A, B > 0 \). Let \( L_i \in \text{End}_A^*(\mathcal{H}, \mathcal{U}_i) \) where \( \mathcal{U}_i \) is a Hilbert C*-module for each \( i \in I \). Suppose that there exist \( C, D > 0 \), such that

\[
C \langle x_i, x_i \rangle_A \leq \langle L_i x_i, L_i x_i \rangle_A \leq D \langle x_i, x_i \rangle_A \quad \text{for all} \quad i \in I, \ x_i \in \mathcal{H}_i
\]

Then,

(a) The sequence \( \{L_i \Lambda_i K \in \text{End}_A^*(\mathcal{H}, \mathcal{U}_i) : i \in I\} \) is a \( K^* \)-\( g \)-frame.

(b) If \( K \) is invertible, then the sequence \( \{L_i \Lambda_i K \in \text{End}_A^*(\mathcal{H}, \mathcal{U}_i) : i \in I\} \) is a \( g \)-frame.
Proof. (a) For all \( x \in \mathcal{H} \), we have

\[
A\langle x, x \rangle_{\mathcal{A}} \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \leq B\langle x, x \rangle_{\mathcal{A}}.
\]

On one hand we have

\[
\sum_{i \in I} \langle L_i \Lambda_i K x, L_i \Lambda_i K x \rangle_{\mathcal{A}} \leq D \sum_{i \in I} \langle \Lambda_i K x, \Lambda_i K x \rangle_{\mathcal{A}} \\
\leq DB\langle K x, K x \rangle_{\mathcal{A}} \\
\leq DB\|K\|^2\langle x, x \rangle_{\mathcal{A}}
\]

One the other hand, we have,

\[
\sum_{i \in I} \langle L_i \Lambda_i K x, L_i \Lambda_i K x \rangle_{\mathcal{A}} \geq C \sum_{i \in I} \langle \Lambda_i K x, \Lambda_i K x \rangle_{\mathcal{A}} \\
\geq CA\langle K x, K x \rangle_{\mathcal{A}} \\
= CA\langle (K^*)^* x, (K^*)^* x \rangle_{\mathcal{A}},
\]

which ends the proof.

(b) Let \( K \) be an invertible operator, we have for all \( x \in \mathcal{H} \)

\[
\langle x, x \rangle_{\mathcal{A}} = \langle K^{-1} K x, K^{-1} K x \rangle_{\mathcal{A}} \\
\leq \|K^{-1}\|^2\langle K x, K x \rangle_{\mathcal{A}} \\
\leq \frac{1}{A}\|K^{-1}\|^2 \sum_{i \in I} \langle \Lambda_i K x, \Lambda_i K x \rangle_{\mathcal{A}} \\
\leq \frac{1}{AC}\|K^{-1}\|^2 \sum_{i \in I} \langle L_i \Lambda_i K x, L_i \Lambda_i K x \rangle_{\mathcal{A}}.
\]

So,

\[
AC\|K^{-1}\|^{-2}\langle x, x \rangle_{\mathcal{A}} \leq \sum_{i \in I} \langle L_i \Lambda_i K x, L_i \Lambda_i K x \rangle_{\mathcal{A}},
\]

which shows that \( \{L_i \Lambda_i K \in \text{End}^*_A(\mathcal{H}, \mathcal{U}_i) : i \in I\} \) is a \( g \)-frame with bounds \( AC\|K^{-1}\|^{-2} \) and \( DB\|K\|^2 \).

\[\square\]

**Definition 3.3.** [10] Let \( \Lambda = \{\Lambda_i\}_{i \in I} \) be a sequence in \( \text{End}^*_A(\mathcal{H}, \mathcal{H}_i) \) for all \( i \in I \)

1. If the \( \mathcal{A} \)-linear hull of \( \bigcup_{i \in I} \Lambda^*(\mathcal{H}_i) \) is dense in \( \mathcal{H} \), then \( \{\Lambda_i\}_{i \in I} \) is \( g \)-complete.
(2) If \( \{ \Lambda_i \}_{i \in I} \) is \( g \)-complete and there exist \( A, B > 0 \) such that for any subset \( J \subseteq I \) and \( y_i \in \mathcal{H}_i \) we have

\[
A \left\| \sum_{j \in J} |y_j|^2 \right\| \leq \left\| \sum_{j \in J} \Lambda_i^* y_j \right\|^2 \leq B \left\| \sum_{j \in J} |y_j|^2 \right\|
\]

then \( \{ \Lambda_i \}_{i \in I} \) is a modular \( g \)-Riesz basis for \( \mathcal{H} \) with respect to \( \{ \mathcal{H}_i \}_{i \in I} \). \( A \) and \( B \) are called bounds of \( \{ \Lambda_i \}_{i \in I} \).

**Theorem 3.4.** Let \( \{ \Lambda_i \in \text{End}_A^x(\mathcal{H}, \mathcal{H}_i), i \in I \} \) and let \( \{ x_{i,j} \}_{j \in J_i} \) be a Parseval frame for \( \mathcal{H}_i \) for each \( i \in I \). Then the following assertions hold

1. The sequence \( \{ \Lambda_i \in \text{End}_A^x(\mathcal{H}, \mathcal{H}_i), i \in I \} \) is a \( g \)-frame (\( g \)-Bessel sequence) in Hilbert \( A \)-module \( \mathcal{H} \) with respect to \( \{ \mathcal{H}_i \}_{i \in I} \) if and only if the sequence \( \left\{ \langle \Lambda_i^* x_{i,j} : i \in I, j \in J_i \right\} \) is a frame in \( \mathcal{H} \) (Bessel sequence).
2. If \( \left\{ \langle \Lambda_i^* x_{i,j} : i \in I, j \in J_i \right\} \) is a modular Riesz basis, then \( \{ \Lambda_i \}_{i \in I} \) is a modular \( g \)-Riesz basis. Conversely if \( \{ \Lambda_i \}_{i \in I} \) is a modular \( g \)-Riesz basis and there exist \( m > 0 \) such that for each \( i \in I \) and \( \{ c_{i,j} \}_{j \in J_i} \) for each finite \( I_1 \subseteq J_i , 

\[
m \left\| \sum_{i \in S} \sum_{j \in J_i} c_{i,j} c_{i,j}^* \right\|^{\frac{1}{2}} \leq \left\| \sum_{i \in S} \sum_{j \in J_i} c_{i,j} \Lambda_i^* x_{i,j} \right\|
\]

then \( \left\{ \langle \Lambda_i^* x_{i,j} : i \in I, j \in J_i \right\} \) is a modular Riesz basis.

**Proof.** (1) Let \( x \in \mathcal{H} \), for each \( i \in I \) we have

\[
\langle \Lambda_i x, \Lambda_i x \rangle_A = \sum_{j \in J_i} \langle \Lambda_i x, x_{i,j} \rangle_A \langle x_{i,j}, \Lambda_i x \rangle_A = \sum_{j \in J_i} |\langle x, \Lambda_i^* x_{i,j} \rangle_A^2 | \langle \Lambda_i^* x_{i,j}, x \rangle_A.
\]

This last equality allows us to conclude that \( \{ \Lambda_i \}_{i \in I} \) is a \( g \)-frame if and only if \( \{ \Lambda_i^* x_{i,j} \}_{i \in I, j \in J_i} \) is a frame.

(2) Let \( \left\{ \langle \Lambda_i^* x_{i,j} : i \in I, j \in J_i \right\} \) be a modular Riesz basis with bounds \( A \) and \( B \). For each \( y_i \in \mathcal{H}_i \) we have

\[
y_i = \sum_{j \in J_i} \langle y_i, x_{i,j} \rangle_A x_{i,j} \text{ and } \langle y_i, y_i \rangle_A = \sum_{j \in J_i} \langle y_i, x_{i,j} \rangle_A \langle x_{i,j}, y_i \rangle_A = \sum_{j \in J_i} |\langle y_i, x_{i,j} \rangle_A|^2 .
\]

Furthermore, we have

\[
\Lambda_i^* y_i = \sum_{j \in J_i} \langle y_i, x_{i,j} \rangle_A \Lambda_i^* x_{i,j}.
\]
So, let $S \subseteq I$ a finite subset, we have
\[
A \| \sum_{i \in S} \langle y_i, y_i \rangle_A \| = A \| \sum_{i \in S} \sum_{j \in J_i} |\langle y_i, x_{i,j} \rangle_A|^2 \|
\leq \| \sum_{i \in S} \sum_{j \in J_i} \langle y_i, x_{i,j} \rangle_A \Lambda_i^* y_i \|^2
= \| \sum_{i \in S} \Lambda_i^* y_i \|^2
\leq B \| \sum_{i \in S} \sum_{j \in J_i} |\langle y_i, x_{i,j} \rangle_A|^2 \|
= B \| \sum_{i \in S} \langle y_i, y_i \rangle_A \|
\]

Conversely, we assume that $\{\Lambda_i\}_{i \in I}$ be a modular $g$-Riesz basis for $\mathcal{H}$ with bounds $A$ and $B$, it follows that for any finite subset $S \subseteq I$,
\[
A \| \sum_{i \in S} \langle y_i, y_i \rangle_A \| \leq \| \sum_{i \in S} \Lambda_i^* y_i \|^2 \leq B \| \sum_{i \in S} \langle y_i, y_i \rangle_A \|
\]
Since $y_i = \sum_{j \in J_i} c_{i,j} x_{i,j}$ and
\[
\| \sum_{i \in S} \langle y_i, y_i \rangle_A \|^2 = \| \sum_{i \in S} \sum_{j \in J_i} c_{i,j} x_{i,j} \Lambda_i^* \|^2
= \| \sum_{i \in S} \sum_{j \in J_i} \langle y_i, x_{i,j} \rangle_A c_{i,j} \|^2
\leq \| \sum_{i \in S} \sum_{j \in J_i} |\langle y_i, x_{i,j} \rangle_A|^2 \| \| \sum_{i \in S} \sum_{j \in J_i} c_{i,j} \|^2
= \| \sum_{i \in S} \langle y_i, y_i \rangle_A \| \| \sum_{i \in S} \sum_{j \in J_i} c_{i,j} \|^2,
\]
then for each $i \in I$, on one hand, we have
\[
\| \sum_{i \in S} \langle y_i, y_i \rangle_A \| \leq \| \sum_{i \in S} \sum_{j \in J_i} c_{i,j} \|^2.
\]
On the other hand, from (3.2) we have
\[
\| \sum_{i \in S} \Lambda_i^* y_i \|^2 = \| \sum_{i \in S} \sum_{j \in J_i} c_{i,j} \Lambda_i^* x_{i,j} \|^2 \leq B \| \sum_{i \in S} \langle y_i, y_i \rangle_A \| \leq \| \sum_{i \in S} \sum_{j \in J_i} c_{i,j} \|^2.
\]
Which ends the proof. □
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