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# Some Mathematical Properties for Kumaraswamy Fréchet distribution

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**Abstract**

In this research, some mathematical properties for Kumaraswamy Fréchet distribution was presented, include entropy, the Shannon entropy, probability weighted moments, moments of residual life and mean of residual life. the properties were concluded for the Kumaraswamy Fréchet distribution using the probability density function (pdf) and cumulative distribution function according to linear representations.

**Keywords:** Probability weighted moments, entropy, Shannon entropy, moment of residual life, mean of residual life.

## 1. INTRODUCTION AND PRLIMINARIES

Generalized distributions are very important in the scope of probability distribution, and it contains many mathematical properties that make the distribution more elastic. In this chapter the definitions and new properties of Kumaraswamy Fréchet distribution are provided [11] in 1972 . The baseline of generalized distributions of Kumaraswamy distribution depends on the probability density function (pdf) and cumulative distribution function as the equation (1.1) and (1.2) respectively are given by:

$$f(x)=ab\, g(x)\, G(x)^{a-1}\, [1-G(x)^{a}]^{b-1} \tag{1.1}$$

where g(x) is density function for distribution and G(x) corresponding cumulative function for the base line.

$$F(x) = 1-[1-G(x)^{a}]^{b} \tag{1.2}$$

$$\text{Where } a>0 \quad , b>0$$

**Definition 1.1.** *Fréchet Distribution*

The Fréchet distribution was introduced in 1927, Fréchet distribution known as inverse Weibull distribution is a special case of the generalizes extreme value distribution. Where the Fréchet distribution was used model maximum values in a data set**.** It is one of four extreme value distribution “EVDs” in common use. The other three are the Gumbel distribution, the Weibull Distribution and the Generalized Extreme Value Distribution [17] in 2018. This distribution is used to model a wide range of phenomena like flood analysis, horse racing, human lifespans, maximum rainfalls, and river discharges in hydrology. Equations (1.3) and (1.4) respectively are represent the probability density function (pdf) and cumulative distribution function (cdf) of the Fréchet distribution with parameter α, β for $x \geq 0$.

$$g(x,\alpha,\beta)=\beta\alpha^{\beta}x^{-\beta-1}exp[-(+\frac{\alpha}{\beta})^{\beta}] \quad (1.3)$$

$$G(x,\alpha,\beta)=exp(-(\frac{\alpha}{x})^{\beta}) \quad (1.4)$$

*Where $\alpha>0$ is scale parameter,*

*and $\beta>0$ is shape parameter*

**Definition 1.2.** The Kumaraswamy Fréchet (Kw-Fr) distribution

By inserting equations (1.3) and (1.4) in equation (1.1) we can find the probability density function (pdf) of Kw-Fr distribution as represented in equation (1.5).

$$f(x,\alpha,\beta,a,b)= ab\,\beta\alpha^{\beta}x^{-\beta-1}exp[-(\frac{\alpha}{x})^{\beta}]\,exp[(-(\frac{\alpha}{x})^{\beta})]^{a-1}[1-\{exp[-(\frac{\alpha}{x})^{\beta}]\}^{a})]^{b-1}$$

then the (pdf) of kw-fr given by:

$$f(x,\alpha,\beta,a,b)= \frac{ab}{x}\,(\frac{\alpha}{x})^{\beta}\,exp[-(\frac{\alpha}{x})^{\beta}]\,exp[(-(\frac{\alpha}{x})^{\beta})]^{a-1}\,[1-\{exp[-a(\frac{\alpha}{x})^{\beta}])]^{b} \quad (1.5)$$

the following graph represents pdf of Kumaraswamy Fréchet (Kw-Fr) distribution by using R program

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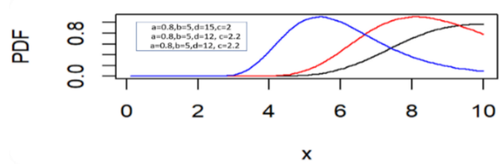


Figure:(1) the pdf for (KW-fr) at different values

Also by inserting equation (1.4) in equation (1.2) we can find the cumulative distribution function (cdf) of Kw-Fr distribution as represented in equation (1.6).

Then the (cdf) of (KW-Fr) is given by

$$F(x,\alpha,\beta,a,b)= 1-[1-exp[-(\frac{\alpha}{x})^{\beta}])^{a}]^{b} \quad (1.6)$$

the following graph represents cdf of Kumaraswamy Fréchet (Kw-Fr) distribution by using R program.

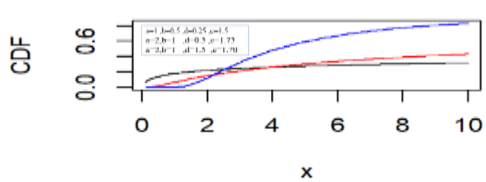


Figure:(2) the cdf for (KW-fr) at different values

# 2. MAIN RESULTS

### *2.1linear representation*

In this subsection the researcher used the following power series expansion that given by the following formula to find the linear representation of Kumaraswamy Fréchet distribution [5] in 2009.

$$(1-y)^c = \sum_{i=0}^{\infty}(-1)^i \binom{c}{i} (y)^i$$

To make pdf of Kumaraswamy Fréchet "kw-fr" in a linear representation form to be able to use in several properties, so we have.

$$f(x,\alpha,\beta,a,b)= \frac{ab}{x} (\frac{\alpha}{x})^{\beta} \ exp[-(\frac{\alpha}{x})^{\beta} ]exp[-(a-1)(\frac{\alpha}{x})^{\beta})][1-\{ exp[-a(\frac{\alpha}{x})^{\beta}]\}]^{b-1}$$

then transfer the last term of the equation to linear representation as the following.

$$[1-\{exp[-a(\frac{\alpha}{x})^{\beta}]\}]^{b-1} = \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} (exp[-a (\frac{\alpha}{x})^{\beta}])^i$$
$$=\sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} (exp\ [-ai(\frac{\alpha}{x})^{\beta}])$$

Therefore, we can obtain the pdf of KW-fr as the following

$$f(x,\alpha,\beta,a,b)= \frac{ab}{x} (\frac{\alpha}{x})^{\beta} exp[-(\frac{\alpha}{x})^{\beta} ]exp[-(a-1)(\frac{\alpha}{x})^{\beta})]\sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} (exp[-ai(\frac{\alpha}{x})^{\beta}]$$
$$f(x,\alpha,\beta,a,b)= \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} \frac{ab}{x} (\frac{\alpha}{x})^{\beta} \ exp[-(\frac{\alpha}{x})^{\beta}-(a-1)(\frac{\alpha}{x})^{\beta}-ai(\frac{\alpha}{x})^{\beta}]$$

*by simplify*

$$f(x, \alpha,\beta,a,b)= \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} \frac{ab}{x} (\frac{\alpha}{x})^{\beta} \ exp[-(\frac{\alpha}{x})^{\beta}(1+(a-1)-ai)]$$
$$f(x, \alpha,\beta,a,b)= \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} \frac{ab}{x} (\frac{\alpha}{x})^{\beta} \ exp[-(\frac{\alpha}{x})^{\beta}(a)(1-i)]$$
$$f(x,\alpha,\beta,a,b)= \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} \frac{ab}{x} (\frac{\alpha}{x})^{\beta} \ exp[+(i-1)a(\frac{\alpha}{x})^{\beta} ]$$

*suppose that* $\daleth_i = \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} ab$

*therefore, the pdf of kw-fr by using linear representation is*

$$f(x,\alpha,\beta,a,b)= \sum_{i=0}^{\infty} \frac{\daleth_i}{x} (\frac{\alpha}{x})^{\beta} \ exp[a(i-1)(\frac{\alpha}{x})^{\beta} ] \qquad (2.1)$$

### *2.2 Probability weighted moments*

Probability weighted moments (PWMs) are introduced and shown to be potentially useful in expressing the parameters of these distributions [12] in 2019 . This method is considered as a generalization of the traditional moments of a probability distribution for a random variable X. The probability weighted moments of a random variable expressible in an inverse form is defined by:

$$\rho_{s,r} = E(x^s F(x)^r = \int_0^{\infty} x^s F(x)^r f(x)dx$$

Where s,r are real numbers and x(F) is the inverse cumulative distribution function. Moreover, the PWMs of a Kumaraswamy Fréchet take the following form:

By substitution with pdf of kw-fr distribution at (1.6) get

$$F(x)^r = [\ 1-[1-\ exp[-(\frac{\alpha}{x})^{\beta}])^a]^b\ ]^r$$
$$\square F(x)^r = [\ 1-[1-\ exp[-a(\frac{\alpha}{x})^{\beta}]^b\ ]^r$$

By using factorial rule *get* $[1-\ exp[-a(\frac{\alpha}{x})^{\beta}]^b$ as the following

$$[1-\ exp[-a(\frac{\alpha}{x})^{\beta}]^b=\sum_{\lambda=0}^{\infty}(-1)^{\lambda} \binom{b}{\lambda} (exp[-a(\frac{\alpha}{x})^{\beta})^{\lambda}$$
$$\square [1-\ exp[-a(\frac{\alpha}{x})^{\beta}]^b=\sum_{\lambda=0}^{\infty}(-1)^{\lambda} \binom{b}{\lambda} (exp[-\lambda a(\frac{\alpha}{x})^{\beta}]$$

Therefore, $F(x)^r$ can represented by

$$F(x)^r = [1-\sum_{\lambda=0}^{\infty}(-1)^{\lambda} \binom{b}{\lambda} (exp\ [-\lambda a(\frac{\alpha}{x})^{\beta}]]^r$$
$$\square F(x)^r = \sum_{l=0}^{\infty}(-1)^l \binom{r}{l} (\sum_{\lambda=0}^{\infty}(-1)^{\lambda} \binom{b}{\lambda})^l \ exp\text{-}\lambda la(\frac{\alpha}{x})^{\beta})$$

suppose that $\sigma\lambda l = (\sum_{\lambda=0}^{\infty}(-1)^{\lambda}\binom{b}{\lambda}))^{l}\ \sum_{l=0}^{\infty}(-1)^{l}\binom{r}{l}$

$$\square F(x)^{r} = \sigma\lambda l\ exp(-\lambda la(\frac{\alpha}{x})^{\beta})$$

Therefore, $F(x)^{r}f(x)\ given\ by$

$$F(x)^{r}f(x) = \sigma\lambda l\ exp\text{-}\lambda la(\frac{\alpha}{x})^{\beta})\sum_{i=0}^{\infty}\frac{Ʈ_i}{x}\ (\frac{\alpha}{x})\beta\ \ exp[-a(1-i)(\frac{\alpha}{x})\beta\ ]$$

$$\square F(x)^{r}f(x) = \sigma\lambda l\ \sum_{i=0}^{\infty}\frac{Ʈ_i}{x}\ (\frac{\alpha}{x})\beta\ \ exp[\text{-}\lambda la(\frac{\alpha}{x})^{\beta}-a(1-i)(\frac{\alpha}{x})\beta]$$

$$\square F(x)^{r}f(x) = \sigma\lambda l\ \sum_{i=0}^{\infty}\ Ʈ_i\alpha^{\beta}x^{-1-\beta}\ exp\text{-}[\lambda la + a(1-i)](\frac{\alpha}{x})^{\beta}$$

$\rho_{s,r}$ get by the following

$$\rho_{s,r} = E(x^{s}F(x)^{r} = \int_{0}^{\infty} x^{s}F(x)^{r}f(x)dx$$

then

$$\rho_{s,r} = E(x^{s}F(x)^{r} = \int_{0}^{\infty} x^{s}\sigma\lambda l\ \sum_{i=0}^{\infty}\ Ʈ_i\alpha^{\beta}x^{-1-\beta}\ \ exp - [\lambda la + a(1-i)](\frac{\alpha}{x})^{\beta}\ dx$$

Let ɯ▯ = $\sigma\lambda l\ \sum_{i=0}^{\infty}\ Ʈ_i\alpha^{\beta}$

$$\square\rho_{s,r} = \square\int_{0}^{\infty} x^{s-\beta-1}\ exp[-a(\lambda l + (1-i)](\frac{\alpha}{x})^{\beta}\ dx$$

and

$$Let\ u = a(\lambda l + (1-i)](\frac{\alpha}{x})^{\beta}$$

$$\square du = a\ (1\text{-}i)\alpha^{\beta}(-\beta)x^{-\beta-1}\ dx$$

$$\square dx = \frac{x^{\beta+1}\ du}{a(1-i)\alpha^{\beta}(-\beta)}$$

$but\ \frac{u}{a} - \lambda l = (\frac{\alpha}{x})^{\beta}, thus$

$$\frac{u-\lambda la}{a(1-i)} = (\frac{\alpha}{x})^{\beta}$$

$$\square\frac{u-\lambda la}{a(1-i)\alpha^{\beta}} = x^{-\beta}$$

Therefore $x^{\beta}\ represented$ by

$$x^{\beta} = \frac{a(1-i)\alpha^{\beta}}{u-\lambda la}$$

$$\square x = (a(1-i))^{\frac{1}{\beta}}\alpha\ (-\lambda la)^{\frac{-1}{\beta}}(1\text{-}\frac{u}{\lambda la})^{\frac{-1}{\beta}}$$

To simplify

$$x = (a(1-i))^{\frac{1}{\beta}}\alpha\ (-\lambda la)^{\frac{-1}{\beta}}\ \sum_{q=0}^{\infty}(q+1)(\frac{u}{\lambda la})^{q}$$

Suppose that $k_2$ given by

$$k_2 = (a(1-i))^{\frac{1}{\beta}}\alpha\ (-\lambda la)^{\frac{-1}{\beta}}\ \sum_{q=0}^{\infty}(q+1)$$

$$\square x = k_2\,u^{q}$$

then

$$dx = \frac{(k2\ u^{q})^{\beta+1}\ du}{a(1-i)\alpha^{\beta}(-\beta)}$$

$$\square dx = \frac{(k2\ )^{\beta+1}}{a(1-i)\alpha^{\beta}(-\beta)}\ u^{q\beta+q}du$$

Therefore, by substitution find Probability weighted moments given by

$$\rho_{s,r}\ = \square\int_{0}^{\infty} {k_2}^{s-\beta-1}\ u^{qs-q\beta-q}\ exp(\ (-u)\ \frac{(k2\ )^{\beta+1}}{a(1-i)\alpha^{\beta}(-\beta)}\ u^{q\beta+q}du$$

$$\square\rho_{s,r} = \frac{\square {k_2}^{s}}{a(1-i)\alpha^{\beta}(-\beta)}\int_{0}^{\infty}\ u^{qs}\ exp\ (-u)\ du$$

Thus, Probability weighted moments of kw-fr distribution given by

$$\rho_{s,r} = \frac{\square {k_2}^{s}}{a(1-i)\alpha^{\beta}(-\beta)}\ du\ \Gamma(qs+1) \qquad (2.2)$$

## *2.3 Entropy*

Entropy is one of the most popular measures of uncertainty. It is used to measure the randomness of system; also, it is a measure of information [17] in 2018As former mathematical work, statistical entropy was introduced by Shannon in 1948 as a basic concept in information theory measuring the amount of information in a random variable X. Entropy function represented by the following:

$$I_\theta(x) = \frac{log}{1-\theta}\int_{-\infty}^{\infty}(f(x))^\theta \, dx \qquad \theta > 0 \neq 1$$

By using pdf of kw-fr at (2.1) to get $(f(x))^\theta$ *as the following*

$$(f(x))^\theta = \sum_{i=0}^{\infty}(\frac{Ꚙ_i}{x})^\theta \ (\frac{\alpha}{x})^{\beta\theta} \ exp\ [a(i-1)(\frac{\alpha}{x})^\beta\ ]^\theta$$

then

$$(f(x))^\theta = (\sum_{i=0}^{\infty} Ꚙ_i \ \alpha^\beta\ )^\theta \ x^{-\beta\theta-\theta} \ exp\ [a\ \theta\ (i-1)(\frac{\alpha}{x})^\beta]$$

Thus, using $(f(x))^\theta$ *to substitute in* Entropy function to get the following

$$I_\theta(x) = \frac{log}{1-\theta}\int_{-\infty}^{\infty}(\sum_{i=0}^{\infty} Ꚙ_i \ \alpha^\beta\ )^\theta \ x^{-\beta\theta-\theta} \ exp\ [a\ \theta\ (i-1)(\frac{\alpha}{x})^\beta]\ dx$$

*By collecting x in the integration to get the following*

$$I_\theta(x) = \frac{(\sum_{i=0}^{\infty} Ꚙ_i \ \alpha^\beta\ )^\theta}{1-\theta}\ log\int_{-\infty}^{\infty}(\ x^{-\theta(\beta+1)} \ exp\ [-a\ \theta\ (1-i)(\frac{\alpha}{x})^\beta]\ dx$$

By separating the previous function into u, x and dx to find the following

Let $u = a\ \theta\ (1-i)(\frac{\alpha}{x})^\beta$

$$\square \frac{u}{a\ \theta\ (1-i)} = \frac{\alpha^\beta}{x^\beta}$$

*To get x*

$$x^\beta = \frac{\alpha^\beta\ a\ \theta\ (1-i)}{u}$$

*then*

$$x = (\alpha^\beta\ a\ \theta\ (1-i))^{\frac{1}{\beta}} \ u^{\frac{-1}{\beta}}$$

$$\square x = \alpha(\ a\ \theta\ (1-i))^{\frac{1}{\beta}} \ u^{\frac{-1}{\beta}}$$

by using derivative with respect to x To find dx

$$dx = \alpha(\ a\ \theta\ (1-i))^{\frac{1}{\beta}}\ (-\frac{1}{\beta})\ u^{-\frac{1}{\beta}-1}\ du$$

$$\square dx = \frac{-\alpha(\ a\ \theta\ (1-i))^{\frac{1}{\beta}}}{\beta}\ \ u^{\frac{-1-\beta}{\beta}}\ du$$

Therefore, by substitution with u ,x and dx in entropy function as the following

$$I_\theta(x) = \frac{(\sum_{i=0}^{\infty} Ꚙ_i \ \alpha^\beta\ )^\theta}{1-\theta}\ log\int_{-\infty}^{\infty}(\ x^{-\theta(\beta+1)} \ exp\ [-a\ \theta\ (1-i)(\frac{\alpha}{x})^\beta]\ dx$$

$$\square I_\theta(x) = \frac{(\sum_{i=0}^{\infty} Ꚙ_i \ \alpha^\beta\ )^\theta}{1-\theta}\ log\int_{-\infty}^{\infty}(\ \alpha(\ a\ \theta\ (1-i))^{\frac{1}{\beta}}\ u^{\frac{-1}{\beta}})^{-\theta(\beta+1)} \ exp\ (-u)\ \frac{-\alpha(\ a\ \theta\ (1-i))^{\frac{1}{\beta}}}{\beta}\ \ u^{\frac{-1-\beta}{\beta}}\ du$$

By collecting u in the same integration

$$I_\theta(x) = \frac{(\sum_{i=0}^{\infty} Ꚙ_i \ \alpha^\beta\ )^\theta}{1-\theta}\ \frac{log\ [(-\alpha)(\ a\ \theta\ (1-i))^{\frac{1}{\beta}}\ (\alpha(\ a\ \theta\ (1-i))^{\frac{1}{\beta}}\ u^{\frac{-1}{\beta}})^{-\theta(\beta+1)}]}{\beta}$$

$$\int_{-\infty}^{\infty} u^{\frac{\theta}{\beta}}\ ^{(1+\beta)}\ e^{-u}\ u^{\frac{-1-\beta}{\beta}}\ du$$

$$\square I_\theta(x) = \frac{(\sum_{i=0}^{\infty} Ꚙ_i \ \alpha^\beta\ )^\theta}{1-\theta}\ \frac{log\ [(-\alpha)(\ a\ \theta\ (1-i))^{\frac{1}{\beta}}\ (\alpha(\ a\ \theta\ (1-i))^{\frac{1}{\beta}}\ u^{\frac{-1}{\beta}})^{-\theta(\beta+1)}]}{\beta}$$

$$\int_{-\infty}^{\infty} u^{\frac{\theta+\theta\beta-1-\beta}{\beta}} e^{-u}\, du$$

Therefore, the final form of entropy of kw-fr distribution given by

$$I_\theta(x) = \frac{-(\sum_{i=0}^{\infty} Ꚍ_i\ \alpha^\beta\ )^\theta}{1-\theta} log[\frac{[(\alpha)^{(\frac{-\theta(1+\beta)}{\beta}+1)}(\alpha\,\theta\,(1-i))^{\frac{1}{\beta}-\frac{\theta}{\beta}(1+\beta)}\ (\alpha(\alpha\,\theta\,(1-i))^{\frac{1}{\beta}}\ u^{\frac{-1}{\beta}})^{-\theta(\beta+1)}}{\beta} \Gamma\frac{\theta+\theta\beta-\beta}{\beta} \quad (2.3)$$

***2.4 Shannon entropy***

$$I_S(\theta) = -E(\ln f(x)) = -\int_0^{\infty} log\, f(x).f(x)dx$$

By using equation (2.1) and substitute in Shannon entropy's function get the following.

$$log\, f(x) = log[\sum_{i=0}^{\infty}\frac{Ꚍ_i}{x}\ (\frac{\alpha}{x})^\beta\ exp[a(i\text{-}1)(\frac{\alpha}{x})^\beta\ ]$$

$$\Rightarrow log\, f(x) = log[\sum_{i=0}^{\infty}\ Ꚍ_i - log(x) + \beta\ log(\frac{\alpha}{x}) + a(i\text{-}1)\ (\frac{\alpha}{x})^\beta\ ]$$

then

$$log\, f(x) = log[\sum_{i=0}^{\infty}\ (Ꚍ_i) - log(x) + \beta\ log(\alpha) - \beta\ log(x) + a(i\text{-}1)(\frac{\alpha}{x})^\beta$$

$$Thus\ I_S(\theta)\ given\ by$$

$$\boldsymbol{I_S(\theta)} = f(x)log\sum_{i=0}^{\infty}\ (Ꚍ_i) - f(x)\ log(x) + \beta\ f(x)\ log(\alpha) - \beta f(x)\ log(x) + a(i\text{-}1)\ (\frac{\alpha}{x})^\beta\ f(x) \quad (2.4)$$

To simplify the previous function, suppose that $\log f(x).f(x)$ separated into four parts as the following
First part represented by I

$$I = \int_0^{\infty} f(x)log\sum_{i=0}^{\infty}\ Ꚍ_i)\ dx$$

$$\Rightarrow I = \ log(\sum_{i=0}^{\infty}\ Ꚍ_i)\int_0^{\infty} f(x)\ dx$$

$$I = log(\sum_{i=0}^{\infty}\ Ꚍ_i)$$

Second part represented by II

$$II = -f(x)\ log(x) - \beta f(x)\ log(x)$$

$$\Rightarrow II = -f(x)\ log(x)[1+\beta]$$

$$\Rightarrow II = -(1+\beta)f(x)\ log(x)$$

Add (1) and subtract (1) from x as and transfer the log get the following

$$II = -(1+\beta)f(x)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{n}\binom{n}{m}(x)^m$$

By using the integration get the following

$$II = -(1+\beta)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{n}\binom{n}{m}E(x^m)$$

Using integration rule to find $E(x^m)$ that given by

$$E(x^m) = \int_0^{\infty}\ (x)^m f(x)\ dx$$

By using pdf of kw-fr distribution in equation (1.5) to get $E(x^m)$ as the following

$$E(x^m) = \sum_{i=0}^{\infty} Ꚍ_i\ \alpha^\beta{}_i\int_0^{\infty}\ (\frac{(a(1-i))^{\frac{1}{\beta}}\alpha}{v^{\frac{1}{\beta}}})^{m-1-\beta}\ \ exp[-v\ ]\ \frac{-\alpha}{\beta}\ \ \ (a(1-i))^{\frac{1}{\beta}}\ v^{-(\frac{\beta=1}{\beta})}\ dv$$

By simplify

$$E(x^m) = \sum_{i=0}^{\infty} Ꚍ_i\ \alpha^\beta\ \ (a(1-i))^{\frac{m-1-\beta}{\beta}}\ \ \alpha^{m-1-\beta}(\frac{-\alpha}{\beta})\ (a(1-i)^{\frac{1}{\beta}}\ \ \int_0^{\infty}\ v^{-(\frac{\beta+1}{\beta})}v^{\frac{\beta-m+1}{\beta}}\ \ e^{-v}\, dv$$

$$\Rightarrow E(x^m) = -\sum_{i=0}^{\infty} Ꚍ_i\ \frac{(\alpha)^{\beta+1+m-\beta-1}}{\beta}\ (a(1-i))^{\frac{m-\beta-1+1}{\beta}}\ \int_0^{\infty}\ v^{\frac{-m}{\beta}}e^{-v}\, dv$$

Put $\int_0^{\infty}\ v^{\frac{-m}{\beta}}e^{-v}$ dv is a gamma formula

$$\square\ E(x^m) = -\sum_{i=0}^{\infty} \text{Ʈ}_i \frac{(\alpha)^m}{\beta} (a(1-i))^{\frac{m-\beta}{\beta}}\, \Gamma\left(1-\frac{m}{\beta}\right)$$

Therefore, Second part II given by

$$II = (1+\beta) \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{n+m}}{n} \binom{n}{m} \text{Ʈ}_i \frac{(\alpha)^m}{\beta} (a(1-i))^{\frac{m-\beta}{\beta}}\, \Gamma\left(1-\frac{m}{\beta}\right)$$

Third part represented by III

$$III = \beta\ \log(\alpha) \int_0^{\infty} f(x)$$

As known $\int_0^{\infty} f(x) = 1$ then

$$III = \beta\ \log(\alpha)$$

Fourth part represented by VI

$$VI = \int_0^{\infty} a(1-i) \left(\frac{\alpha}{x}\right)^{\beta} f(x)\, dx$$

In this part substitute with pdf at function (2.1) as the following

$$VI = a(1-i)\alpha^{\beta} \int_0^{\infty} x^{-\beta} \sum_{i=0}^{\infty} \frac{\text{Ʈ}_i}{x} \left(\frac{\alpha}{x}\right) \beta\ \exp\left[a(i-1)\left(\frac{\alpha}{x}\right)^{\beta}\right] dx$$

$$\square VI = a(1-i)\alpha^{2\beta} \sum_{i=0}^{\infty} \text{Ʈ}_i \int_0^{\infty} x^{-2\beta-1} \exp\left[-a(1-i)\left(\frac{\alpha}{x}\right)^{\beta}\right] dx$$

To simplify

Let $u = a(1-i)\alpha^{\beta} x^{-\beta}$

then

$$x^{\beta} = \frac{a(1-i)\alpha^{\beta}}{u}$$

$$\square x = \frac{(a(1-i))^{\frac{1}{\beta}}\alpha}{u^{\frac{1}{\beta}}}$$

$$\square x = (a(1-i))^{\frac{1}{\beta}}\alpha\, u^{\frac{-1}{\beta}}$$

$$\square dx = (a(1-i))^{\frac{1}{\beta}}\alpha \left(\frac{-1}{\beta}\right) u^{\frac{-1}{\beta}-1} du$$

By substitution with dx and use gamma formula get the following

$$VI = \frac{-[a(1-i)]^{-1}}{\beta} \sum_{i=0}^{\infty} \text{Ʈ}_i\ \Gamma(2)$$

Thus, VI given by

$$VI = \frac{-\sum_{i=0}^{\infty} \text{Ʈ}_i}{\beta\, a(1-i)}$$

By substitution with four parts I, II, III and VI in Shannon entropy's function at (2.4) get the following

$$\square \boldsymbol{I_S(\theta) = \log\left(\sum_{i=0}^{\infty} \text{Ʈ}_i\right) + (1+\beta) \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{n+m}}{n} \binom{n}{m} \text{Ʈ}_i \frac{(\alpha)^m}{\beta} (a(1-i))^{\frac{m-\beta}{\beta}}\, \Gamma\left(\frac{m}{\beta}\right) + \beta\ \log(\alpha) + \frac{-\sum_{i=0}^{\infty} \text{Ʈ}_i}{\beta\, a(1-i)}} \qquad (2.5)$$

### *2.5 Moment of residual life*

In life testing situations, the additional lifetime given that component has survived until time t is called the residual life function (RLF) of the component [2] in 2017. Moment of residual represented as the following:

$$m(t)_n = \frac{1}{R(t)} \int_t^{\infty} (x-t)^n f(x)\, dx \qquad (2.6)$$

$$let\ (x-t)^n = (-t)^n\ (1-\frac{x}{t})^n$$

$$□(x-t)^n = (-t)^n \sum_{h=0}^{\infty}(-1)^h \binom{n}{h}(\frac{x}{t})^h$$

By substitution in (2.6) get the following

$$□m(t)_n = \frac{1}{R(t)} \int_t^{\infty} \sum_{h=0}^{\infty}(-1)^h \binom{n}{h}(\frac{x}{t})^h f(x)$$

By simplify

$$m(t)_n = \frac{\sum_{h=0}^{\infty}(-1)^h\binom{n}{h}}{R(t)t^h} \int_t^{\infty} x^h f(x)dx$$

by substitution with pdf of kw-fr at (7) get the following

$$m(t)_n = \frac{\sum_{h=0}^{\infty}(-1)^h\binom{n}{h}}{R(t)t^h} \int_t^{\infty} x^h \sum_{i=0}^{\infty} \frac{\text{Ƭ}_i}{x}\ (\frac{\alpha}{x})^{\beta}\ exp[-ai(\frac{\alpha}{x})^{\beta}\ ]\ dx$$

$$□m(t)_n = \frac{\sum_{h=0}^{\infty}(-1)^h\binom{n}{h}\sum_{i=0}^{\infty}\text{Ƭ}_{i}\,\alpha\beta}{R(t)t^h} \int_t^{\infty} x^{h-\beta-1}\ \ exp[-ai(\frac{\alpha}{x})^{\beta}\ ]\ dx$$

$$Let \qquad u=ai(\frac{\alpha}{x})^{\beta}$$

$$u=ai\ \alpha^{\beta}x^{-\beta}$$

$$du= ai\ \alpha^{\beta}(-\beta)x^{-\beta-1}dx$$

$$x^{-\beta} = \frac{u}{ai\ \alpha^{\beta}}$$

$$x^{\beta} = ai\ \alpha^{\beta}\ u^{-1}$$

$$x= (ai)^{\frac{1}{\beta}}\ \alpha\ u^{\frac{-1}{\beta}}$$

$$dx= (ai)^{\frac{1}{\beta}}\ \alpha\ (\frac{-1}{\beta})u^{\frac{-1}{\beta}-1}\ du$$

$$m(t)_n = \frac{\sum_{h=0}^{\infty}(-1)^h\binom{n}{h}\sum_{i=0}^{\infty}\text{Ƭ}_{i}\,\alpha\beta}{R(t)t^h} \int_t^{\infty} ((ai)^{\frac{1}{\beta}}\ \alpha\ u^{\frac{-1}{\beta}})^{h-\beta-1}\ exp[u]\ \ (ai)^{\frac{1}{\beta}}\ \alpha\ \frac{-1}{\beta}\ u^{\frac{-1}{\beta}-1}\ du$$

$$m(t)_n = \frac{\sum_{h=0}^{\infty}(-1)^h\binom{n}{h}\sum_{i=0}^{\infty}\text{Ƭ}_{i}\,\alpha\beta\ \ (ai)^{\frac{1}{\beta}}\ \alpha\ )^{h-\beta-1}\ (ai)^{\frac{1}{\beta}}\ \frac{-\alpha}{\beta}}{R(t)t^h} \int_t^{\infty} u^{\frac{\beta-h+1}{\beta}}\ \ u^{\frac{-1-\beta}{\beta}}\ \ e^{-u}\ \ du$$

$$Let\ K_3 = \frac{\sum_{h=0}^{\infty}(-1)^h\binom{n}{h}\sum_{i=0}^{\infty}\text{Ƭ}_{i}\,\alpha\beta\ \ (ai)^{\frac{1}{\beta}}\ \alpha\ )^{h-\beta-1}\ (ai)^{\frac{1}{\beta}}\ \frac{-\alpha}{\beta}}{R(t)t^h}$$

$$m(t)_n = K_3\int_t^{\infty} u^{\frac{-h}{\beta}}\ \ e^{-u}\ \ du$$

$$u=ai\ \alpha^{\beta}x^{-\beta}$$

$$u=ai\ \alpha^{\beta}t^{-\beta}$$

$$u=\frac{ai\ \alpha^{\beta}}{t^{\beta}}$$

$$m(t)_n = K_3\int_{\frac{ai\ \alpha\beta}{t^{\beta}}}^{\infty} u^{\frac{-h}{\beta}}\ \ e^{-u}\ \ du$$

$$m(t) = K_3\ \Gamma(1-\frac{h}{\beta},\frac{ai\ \alpha^{\beta}}{t^{\beta}}) \tag{2.7}$$

## 2.6 Mean of residual life

The mean of residual life is an important application of the moments of residual lifetime function which has been employed in life lengths studies by various authors. It has also been shown that the exponential function determines the distribution function uniquely [16] in 2016. It is also well known that a constant MRL function implies that the distribution is exponential and represented as the following:

$$m(t)=\frac{1}{R(t)}\int_t^{\infty} x\, f(x)\ dx - t$$

$$m(t)=\frac{1}{R(t)}\int_t^{\infty} x \ \sum_{i=0}^{\infty}\frac{\daleth_i}{x}\ (\frac{\alpha}{x})^{\beta}\ exp[-a(1-i)(\frac{\alpha}{x})^{\beta}]\ dx\text{-}t$$

$$m(t)=\frac{\alpha^{\beta}\sum_{i=0}^{\infty}\daleth_i}{R(t)}\int_t^{\infty} x^{-\beta}\ \ exp[-a(1-i)(\frac{\alpha}{x})^{\beta}]\ dx\text{-}t$$

thus

$$m(t)=\frac{\alpha^{\beta}\sum_{i=0}^{\infty}\daleth_i}{R(t)}\int_t^{\infty} x^{-\beta}\ \ e^{-a(1-i)(\frac{\alpha}{x})\beta}\ \ dx\text{-}t$$

$$let\ u= a(1-i)(\frac{\alpha}{x})^{\beta}$$

$$du= a(1-i)\alpha^{\beta}\ (-\beta)x^{-\beta-1}\ dx$$

*thus, get dx as the following*

$$dx=\frac{-x^{\beta+1}\ du}{a(1-i)\alpha^{\beta}\beta}$$

*then*

$$x^{\beta}=\frac{a(1-i)\alpha^{\beta}}{u}$$

$$x= a(1-i))^{\frac{1}{\beta}}\alpha u^{\frac{-1}{\beta}}$$

by substitution by the x, dx and du get the following

$$m(t)=\frac{\boldsymbol{\alpha^{\beta}}\sum_{i=0}^{\infty}\daleth_i}{\boldsymbol{R(t)}}\int_{\boldsymbol{t}}^{\infty}(a(1-i))^{\frac{1}{\beta}})^{-\boldsymbol{\beta}}\ \boldsymbol{\alpha^{-\beta}}\ u\ \boldsymbol{e}^{-u}\ \ \frac{-x^{\beta+1}}{a(1-i)\alpha^{\beta}\beta}\ \ du-t$$

$$m(t)=\frac{-\alpha^{\beta}\sum_{i=0}^{\infty}\daleth_i}{a(1-i)\alpha^{\beta}R(t)}\int_t^{\infty}\frac{u\ e^{-u}\,x^{\beta+1}}{a(1-i)\alpha^{\beta}\beta}\ \ du-t$$

*thus*

$$m(t)=\frac{-\boldsymbol{\alpha^{\beta}}\sum_{i=0}^{\infty}\daleth_i}{\boldsymbol{R(t)}\,(a(1-i)\alpha^{\beta})^2\beta}\int_{\boldsymbol{t}}^{\infty}\ u\,x^{\beta+1}\,\boldsymbol{e}^{-u}\ \ du-t$$

$$m(t)=\frac{-\boldsymbol{\alpha^{\beta}}\sum_{i=0}^{\infty}\daleth_i}{\boldsymbol{R(t)}\,(a(1-i)\alpha^{\beta})^2\beta}\int_{\boldsymbol{t}}^{\infty}\ u\,(a(1-i))^{\frac{1}{\beta}}\alpha u^{\frac{-1}{\beta}})^{\beta+1}\,\boldsymbol{e}^{-u}\ \ du-t$$

$$m(t)=\frac{-\alpha^{\beta}\sum_{i=0}^{\infty}\daleth_i}{R(t)\,(a(1-i)\alpha^{\beta})^2\beta}\int_t^{\infty}\ u(a(1-i))^{\frac{\beta+1}{\beta}}\alpha^{\beta+1}u^{\frac{-\beta-1}{\beta}}\,e^{-u}\ \ du-t$$

by collecting u and $e^{-u}$ in the integration as gamma form, get the following

$$m(t)=\frac{-\alpha^{\beta}\sum_{i=0}^{\infty}\daleth_i\,(a(1-i))^{\frac{\beta+1}{\beta}}\alpha^{\beta+1}}{R(t)\,(a(1-i)\alpha^{\beta})^2\beta}\int_t^{\infty}\ u^{\frac{-1}{\beta}+1}\,e^{-u}\ \ du-t$$

*thus*

$$m(t)=\frac{-\sum_{i=0}^{\infty}\daleth_i\,\alpha^{2\beta+1-2\beta}\ (a(1-i))^{\frac{\beta+1}{\beta}-2}}{\beta\,R(t)}\ \ \Gamma\left(2-\frac{1}{\beta}\right)-t$$

therefore, the final form of Moments of Residual Life represented by

$$m(t)=\frac{-\sum_{i=0}^{\infty}\daleth_i\,\alpha\ (a(1-i))^{\frac{1-\beta}{\beta}}}{\beta\,R(t)}\ \ \Gamma\left(\frac{2\beta-1}{\beta}\right)-t \qquad (2.8)$$

## 4. CONCLUSION

In this research paper, we introduce some mathematical properties of the Kumaraswamy Fréchet distribution based on linear representation. These properties are probability weighted moments, entropy, Shannon entropy, moment of residual life and Mean of Residual. The introduced mathematical properties are new for Kumaraswamy Fréchet distribution. Therefore, we invite researchers to study more mathematical properties of the distributions because of its many applications which can contribute to solving many life problems.

## CONFLICT OF INTEREST

There is no conflict of interest.

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