An analog of Hardy’s theorem for the second Hankel-Clifford transform

1Mohamed El Hamma, 2Radouan Daher, 3Hassnaa Lahmadi

1,2,3Laboratoire Mathématiques Fondamentales et appliquées, Faculté des Sciences Aïn Chock, Université Hassan II, B.P 5366 Maarif, Casablanca, Maroc.

Email address: 1m_elhamma@yahoo.fr, 2rj024daher@gmail.com, 3hasnaa.lahmadi@gmail.com

Abstract

In this work, we generalize theorem of Hardy for the second Hankel-Clifford transform.

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1 Introduction and preliminaries

Hardy established in [4], the following theorem

Theorem 1.1 Let \( f \) is measurable function on \( \mathbb{R} \) such that

1. \( |f(x)| \leq Ce^{-Ax^2} \), for all \( x \in \mathbb{R} \)
2. \( |\hat{f}(y)| \leq Ce^{-By^2} \), for all \( y \in \mathbb{R} \),

where where \( A, B, C \) are positive constants.

If \( AB > \frac{1}{4} \), then \( f = 0 \) almost everywhere.

If \( AB < \frac{1}{4} \), then there are infinitely many linearly independent functions satisfying 1) and 2).

If \( AB = \frac{1}{4} \), then the function \( f \) is a constant multiple of \( e^{-Ax^2} \), i.e: \( f(x) = ke^{-Ax^2} \) for some \( k \), where \( \hat{f} \) stands for the Fourier transform of \( f \).
The main of this paper is to establish an analog of this theorem for the second Hankel-Clifford transform. There many analogues of this theorem: for the Jacobi-Dunkl transform, for the Bessel-Struve transform and etc (for example, see [3, 7]).

Now, we collect some basic facts about the second Hankel-Clifford transform. Main references are [2, 8, 10].

We consider the differential-difference operator $D_\mu$ on $(0, +\infty)$, given by

$$D_\mu = x \frac{d^2}{dx^2} + (\mu + 1) \frac{d}{dx},$$

where $\mu \geq 0$.

The Bessel-Clifford function $c_\mu$ of the first kind of order $\mu$ such that $D_\mu c_\mu(x) = -c_\mu(x)$. Given by ([6])

$$c_\mu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k x^k}{k! \Gamma(\mu + k + 1)},$$

where $\Gamma(x)$ is the gamma-function.

From the Bessel function of the first kind $J_\mu(x)$, we have

$$J_\mu(2x^{1/2}) = x^{\mu/2} c_\mu(x),$$

i.e.,

$$c_\mu(x) = x^{-\mu/2} J_\mu(2x^{1/2}),$$

where the Bessel function $J_\mu$ was defined in [4] by

$$2^\mu x^{-\mu} J_\mu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \gamma(\mu + k + 1)} \left( \frac{x}{2} \right)^{2k}.$$

From [1], we have

$$\sqrt{x} J_\mu(x) = O(1), \ x \geq 0. \quad (1)$$

In the following we denote $I = (0, +\infty)$ the open interval in $\mathbb{R}$. We define $L^p_\mu(I), \ 1 \leq p < \infty$ and $\mu \geq 0$, the space of all measurable functions $f$ on $I$, such that

$$\|f\|_{L^p_\mu(I)} = \left( \int_0^\infty |f(x)|^p x^\mu dx \right)^{1/p} < \infty.$$

The second Hankel-Clifford transform is defined for a function $f \in L^1_\mu(I)$, (see [5, 9]), by
\[ h_{2,\mu}(f)(\lambda) = \int_{0}^{+\infty} c_{\mu}(\lambda x) f(x)x^{\mu} dx. \]

The inversion formula of the second Hankel-Clifford transform is defined by

\[ f(x) = \int_{0}^{+\infty} c_{\mu}(\lambda x) h_{2,\mu}(f)(\lambda) \lambda^{\mu} d\lambda \]

From [9], we have the following relation

\[ h_{2,\mu}^{-1} = h_{2,\mu} \tag{2} \]

### 2 Main result

In this section, we will obtain an analog of Hardy’s theorem for the second Hankel-Clifford transform. Now, we give the main result of this paper.

**Theorem 2.1** Suppose that \( f \) is a measurable function on \( I \) such that

\[ |f(x)| \leq ce^{-ax} \tag{3} \]

and

\[ |h_{2,\mu}(f)(\lambda)| \leq ce^{\frac{-\lambda^2}{a}} \]

for some constants \( a > 0 \) and \( c > 0 \). Then the function \( f \) is a constant of \( e^{-x^2/a} \).

In order to prove Theorem 2.1 we will use the following lemma ([11])

**Lemma 2.2** Let \( h \) be an entire function on \( \mathbb{C} \) such that

\[ \forall w \in \mathbb{C}, \ |h(w)| \leq Ae^{a|w|^2} \tag{4} \]

and

\[ \forall t \in \mathbb{R}, \ |h(t)| \leq Ae^{-at^2} \]

for some positive constants \( a \) and \( A \). Then \( h(w) = \text{cont}e^{-aw^2}, \ w \in \mathbb{C} \).

**Proof of theorem 2.1.** First, \( h_{2,\mu}(f)(\lambda) \) is an entire function on \( \mathbb{C} \). From the following formulas (1), (3) and (4) for all \( \lambda \in \mathbb{C} \), we have
\[ |h_{2,\mu}(f)(\lambda)| = \left| \int_{0}^{+\infty} x^{\mu} c_{\mu}(\lambda x) f(x) \, dx \right| \]
\[ = \left| \int_{0}^{+\infty} x^{\mu} (\lambda x)^{-\frac{\mu}{2}} J_{\mu}(2 \lambda^{\frac{1}{2}} x^{\frac{1}{2}}) f(x) \, dx \right| \]
\[ = \left| \int_{0}^{+\infty} x^{\mu} (\lambda^{\frac{1}{2}} x^{\frac{1}{2}})^{-\mu} J_{\mu}(2 \lambda^{\frac{1}{2}} \sqrt{x}) f(x) \, dx \right| \]
\[ = \left| \int_{0}^{+\infty} x^{\mu} 2^{\mu} (2 \lambda^{\frac{1}{2}} x^{\frac{1}{2}})^{-\mu} J_{\mu}(2 \lambda^{\frac{1}{2}} \sqrt{x}) f(x) \, dx \right| \]
\[ \leq \int_{0}^{+\infty} x^{\mu} \sum_{k=0}^{+\infty} \frac{(2 \lambda^{\frac{1}{2}} \sqrt{x})^{2k}}{k! \Gamma(k + \mu + 1)} f(x) \, dx \]
\[ \leq c \int_{0}^{+\infty} x^{\mu} e^{-ax} \sum_{k=0}^{+\infty} \frac{|\lambda|^k}{k! \Gamma(k + \mu + 1)} x^k \, dx \]
\[ \leq c \sum_{k=0}^{+\infty} \frac{|\lambda|^k}{k! \Gamma(k + \mu + 1)} \int_{0}^{+\infty} x^{\mu+k} e^{-ax} \, dx \]
\[ \leq c \sum_{k=0}^{+\infty} \frac{|\lambda|^k}{k! \Gamma(k + \mu + 1)} \int_{0}^{+\infty} \frac{\mu+k}{a^{\mu+k}} e^{-\frac{1}{a}} \, dt \]
\[ \leq c \sum_{k=0}^{+\infty} \frac{|\lambda|^k}{k! \Gamma(k + \mu + 1)} \frac{\Gamma(\mu + k + 1)}{a^{\mu+k+1}} \]
\[ = c \sum_{k=0}^{+\infty} \frac{|\lambda|^k}{k! a^{\mu+k+1}} \]
\[ = \frac{c}{a^{\mu+1}} \sum_{k=0}^{+\infty} \left( \frac{|\lambda|}{a} \right)^k \]
\[ = \frac{c}{a^{\mu+1}} \frac{1}{e^{\frac{|\lambda|}{a}}} \]

If \(|\lambda| < 1\), then
\[ |h_{2,\mu}(f)(\lambda)| \leq \frac{c}{a^{\mu+1}} e^{\frac{1}{a}} \leq \frac{c}{a^{\mu+1}} e^{\frac{1}{a}} e^{\frac{|\lambda|^2}{a}} \]
we put \(c' = \frac{c}{a^{\mu+1}} e^{\frac{1}{a}}\). Thus, for \(|\lambda| < 1\), we have
\[ |h_{2,\mu}(f)(\lambda)| \leq c' e^{\frac{|\lambda|^2}{a}}. \]
If $|\lambda| \geq 1$, then

$$|h_{2,\mu}(f)(\lambda)| \leq \frac{c}{a^{\mu+1}} e^{\frac{|\lambda|}{a}} \leq \frac{c |\lambda|^2}{a^{\mu+1}} e^{\frac{|\lambda|}{a}}.$$  

Then, for all $\lambda \in \mathbb{C}$, we get

$$|h_{2,\mu}(f)(\lambda)| \leq Ae^{\frac{|\lambda|^2}{a}}$$

we also have

$$|h_{2,\mu}(f)(t)| \leq ce^{-\frac{t^2}{a}}, \text{ for all } t \in \mathbb{R}$$

By Lemma 2.2, we have

$$h_{2,\mu}(f)(\lambda) = \text{cont.} e^{-\frac{\lambda^2}{a}} \text{ for all } \lambda \in \mathbb{C}$$

using (2), we obtain

$$f(x) = \text{cont.} e^{-\frac{x^2}{a}}$$

Now, this ends the proof of our main result, namely Generalization of Hardy’s theorem for the second Hankel-Clifford transform. 

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**References**


