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# An analog of Hardy's theorem for the second Hankel-Clifford transform

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Abstract

In this work, we generalize theorem of Hardy for the second Hankel-Clifford transform.
**Keywords:** Second Hankel-Clifford transformation, Bessel-Clifford function.

## 1 Introduction and preliminaries

Hardy established in [4], the following theorem

**Theorem 1.1** *Let $f$ is measurable function on $\mathbb{R}$ such that*

1. $|f(x)| \leq Ce^{-Ax^2}$, *for all* $x \in \mathbb{R}$
2. $|\widehat{f}(y)| \leq Ce^{-By^2}$, *for all* $y \in \mathbb{R}$,

*where where $A$, $B$, $C$ are positive constants.*

*If $AB > \frac{1}{4}$, then $f = 0$ almost everywhere.*

*If $AB < \frac{1}{4}$, then there are infinitely many linearly independent fuctions satisfying* 1) *and* 2) *.*

*If $AB = \frac{1}{4}$, then the function $f$ is a constant multiple of $e^{-Ax^2}$, i.e: $f(x) = ke^{-Ax^2}$ for some $k$, where $\widehat{f}$ stands for the Fourier transform of $f$.*

The main of this paper is to establish an analog of this theorem for the second Hankel-Clifford transform. There many analogues of this theorem: for the Jacobi-Dunkl transform, for the Bessel-Struve transform and etc (for example, see [3, 7]).

Now, we collect some basic facts about the second Hankel-Clifford transform. Main references are [2, 8, 10].

We consider the differential-difference operator $\mathrm{D}_\mu$ on $(0, +\infty)$, given by

$$\mathrm{D}_\mu = x\frac{d^2}{dx^2} + (\mu+1)\frac{d}{dx},$$

where $\mu \geq 0$.

The Bessel-Clifford function $c_\mu$ of the first kind of order $\mu$ such that $\mathrm{D}_\mu c_\mu(x) = -c_\mu(x)$. Given by ([6])

$$c_\mu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k x^k}{k!\Gamma(\mu+k+1)},$$

where $\Gamma(x)$ is the gamma-function.

From the Bessel function of the first kind $J_\mu(x)$, we have

$$J_\mu(2x^{\frac{1}{2}}) = x^{\frac{\mu}{2}} c_\mu(x),$$

i.e.,

$$c_\mu(x) = x^{\frac{-\mu}{2}} J_\mu(2x^{\frac{1}{2}}),$$

where the Bessel function $J_\mu$ was defined in [4] by

$$2^\mu x^{-\mu} J_\mu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\gamma(\mu+k+1)} \left(\frac{x}{2}\right)^{2k}.$$

From [1], we have

$$\sqrt{x} J_\mu(x) = O(1), \; x \geq 0. \tag{1}$$

In the following we denote $\mathrm{I} = (0, +\infty)$ the open interval in $\mathbb{R}$. We define $\mathrm{L}^p_\mu(\mathrm{I})$, $1 \leq p < \infty$ and $\mu \geq 0$, the space of all measurable functions $f$ on I, such that

$$\|f\|_{\mathrm{L}^p_\mu(\mathrm{I})} = \left(\int_0^\infty |f(x)|^p x^\mu dx\right)^{1/p} < \infty.$$

The second Hankel-Clifford transform is defined for a function $f \in \mathrm{L}^1_\mu(\mathrm{I})$, (see [5, 9]), by

$$h_{2,\mu}(f)(\lambda) = \int_0^{+\infty} c_\mu(\lambda x) f(x) x^\mu dx.$$

The inversion formula of the second Hankel-Clifford transform is defined by

$$f(x) = \int_0^{+\infty} c_\mu(\lambda x) h_{2,\mu}(f)(\lambda) \lambda^\mu d\lambda$$

From [9], we have the following relation

$$h_{2,\mu}^{-1} = h_{2,\mu} \tag{2}$$

# 2 Main result

In this section, we will obtain an analog of Hardy's theorem for the second Hankel-Clifford transform. Now, we give the main result of this paper.

**Theorem 2.1** *Suppose that $f$ is a measurable function on* I *such that*

$$|f(x)| \leq ce^{-ax} \tag{3}$$

*and*

$$|h_{2,\mu}(f)(\lambda)| \leq ce^{\frac{-\lambda^2}{a}}$$

*for some constants $a > 0$ and $c > 0$. Then the function $f$ is a constant of $e^{\frac{-x^2}{a}}$.*

In order to prove Theorem 2.1 we will use the following lemma ([11])

**Lemma 2.2** *Let $h$ be an entire function on $\mathbb{C}$ such that*

$$\forall w \in \mathbb{C}, \ |h(w)| \leq Ae^{a|w|^2} \tag{4}$$

*and*

$$\forall t \in \mathbb{R}, \ |h(t)| \leq Ae^{-at^2}$$

*for some positive constants $a$ and $A$. Then $h(w) = cont.e^{-aw^2}, \ w \in \mathbb{C}$.*

**Proof of theorem 2.1.** First, $h_{2,\mu}(f)(\lambda)$ is an entire function on $\mathbb{C}$. From the following formulas (1), (3) and (4) for all $\lambda \in \mathbb{C}$, we have

$$
\begin{aligned}
|h_{2,\mu}(f)(\lambda)| &= \left|\int_0^{+\infty} x^{\mu} c_{\mu}(\lambda x) f(x)dx\right| \\
&= \left|\int_0^{+\infty} x^{\mu} (\lambda x)^{\frac{-\mu}{2}} J_{\mu}(2\lambda^{\frac{1}{2}} x^{\frac{1}{2}}) f(x)dx\right| \\
&= \left|\int_0^{+\infty} x^{\mu} (\lambda^{\frac{1}{2}} x^{\frac{1}{2}})^{-\mu} J_{\mu}(2\lambda^{\frac{1}{2}} \sqrt{x}) f(x)dx\right| \\
&= \left|\int_0^{+\infty} x^{\mu} 2^{\mu} (2\lambda^{\frac{1}{2}} x^{\frac{1}{2}})^{-\mu} J_{\mu}(2\lambda^{\frac{1}{2}} \sqrt{x}) f(x)dx\right| \\
&\leq \int_0^{+\infty} x^{\mu} \sum_{k=0}^{+\infty} \frac{\left(\frac{2|\lambda|^{\frac{1}{2}}\sqrt{x}}{2}\right)^{2k}}{k!\Gamma(k+\mu+1)} f(x)dx \\
&\leq c\int_0^{+\infty} x^{\mu} e^{-ax} \sum_{k=0}^{+\infty} \frac{|\lambda|^k}{k!\Gamma(k+\mu+1)} x^k dx \\
&\leq c\sum_{k=0}^{+\infty} \frac{|\lambda|^k}{k!\Gamma(k+\mu+1)} \int_0^{+\infty} x^{\mu+k} e^{-ax} dx \\
&\leq c\sum_{k=0}^{+\infty} \frac{|\lambda|^k}{k!\Gamma(k+\mu+1)} \int_0^{+\infty} \frac{t^{\mu+k}}{a^{\mu+k}} e^{-t} \frac{1}{a} dt \\
&\leq c\sum_{k=0}^{+\infty} \frac{|\lambda|^k}{k!\Gamma(k+\mu+1)} \frac{\Gamma(\mu+k+1)}{a^{\mu+k+1}} \\
&= c\sum_{k=0}^{+\infty} \frac{|\lambda|^k}{k! a^{k+\mu+1}} \\
&= \frac{c}{a^{\mu+1}} \sum_{k=0}^{+\infty} \frac{\left(\frac{|\lambda|}{a}\right)^k}{k!} \\
&= \frac{c}{a^{\mu+1}} e^{\frac{|\lambda|}{a}}
\end{aligned}
$$

If $|\lambda| < 1$, then

$$|h_{2,\mu}(f)(\lambda)| \leq \frac{c}{a^{\mu+1}} e^{\frac{1}{a}} \leq \frac{c}{a^{\mu+1}} e^{\frac{1}{a}} e^{\frac{|\lambda|^2}{a}}$$

we put $c' = \frac{c}{a^{\mu+1}} e^{\frac{1}{a}}$. Thus, for $|\lambda| < 1$, we have

$$|h_{2,\mu}(f)(\lambda)| \leq c' e^{\frac{|\lambda|^2}{a}}.$$

If $|\lambda| \geq 1$, then

$$|h_{2,\mu}(f)(\lambda)| \leq \frac{c}{a^{\mu+1}} e^{\frac{|\lambda|}{a}} \leq \frac{c}{a^{\mu+1}} e^{\frac{|\lambda|^2}{a}}.$$

Then, for all $\lambda \in \mathbb{C}$, we get

$$|h_{2,\mu}(f)(\lambda)| \leq A e^{\frac{|\lambda|^2}{a}}$$

we also have

$$|h_{2,\mu}(f)(t)| \leq c e^{\frac{-t^2}{a}}, \ for\ all\ t \in \mathbb{R}$$

By Lemma 2.2, we have

$$h_{2,\mu}(f)(\lambda) = cont.e^{\frac{-\lambda^2}{a}} \ for\ all\ t \in \mathbb{C}$$

using (2), we obtain

$$f(x) = cont.e^{\frac{-x^2}{a}}$$

Now, this ends the proof of our main result, namely Generalization of Hardy's theorem for the second Hankel-Clifford transform. ■

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