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New Mathematical Properties For Rayleigh distribution

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Abstract

Regression analysis is one of the most commonly statistical techniques used for analyzing data in different fields. And used to fit the relation between the dependent variable and the independent variables require strong assumption to be met in the model. Generalized linear models (GLMs) allow the extension of linear modeling ideas to a wider class of response types, such as count data or binary responses. Many statistical methods exist for such data types, but the advantage of the GLM approach is that it unites a seemingly disparate collection of response types under a common modeling methodology. So, the problem of the current research is to try to provide a new mathematical property for Exponentiated Rayleigh distribution, and it was one of the most important properties that was studied is to determine Harmonic Mean, as well as calculating the Quantile function, Moments of Residual life (MRL), Reversed Residual Life, Mean of Residual life. The study also presented the probability density function (pdf) and cumulative distribution function according to linear representations.

Keywords: Harmonic Mean, Moment Generating Function, Characteristic Function, Quantile function, Raw Moments, Moments of Residual life (MRL), Reversed Residual Life, Mean of Residual life.

1. Introduction and Preliminaries

Generalized distributions are important in the scope of probability distribution, and it contains many mathematical properties that make the distribution more elastic. In this chapter the definitions and new properties of Rayleigh distribution are provided. Generalized distributions depend on two things, namely the cumulative distribution function (CDF) and the probability density function (pdf):

$$\mathrm{f(x) = ag(x)}[G(X)]^{a-1} a > 0, x > 0 \qquad (1)$$

(CDF) by integral the probability density function

$$\mathrm{F(x)} = \int_0^x \mathrm{ag(x)}[G(X)]^{a-1}\,\mathrm{ag(x)}[G(X)]^{a-1}dx$$

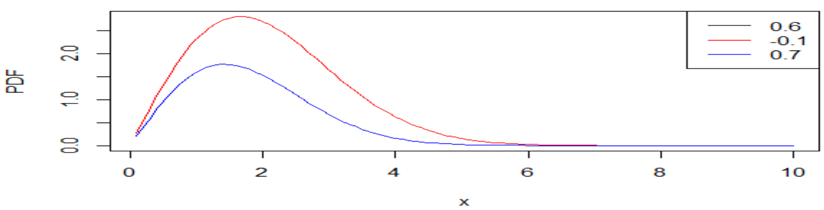
$$\mathrm{F(x)} = [G(X)]^{a} a > 0, x > 0 \qquad (2)$$

where $\boldsymbol{g}(\boldsymbol{x}) = \frac{\boldsymbol{dG}(x)}{\boldsymbol{dx}}$ and an additional positive shape, $G(X)$ is the baseline of distribution

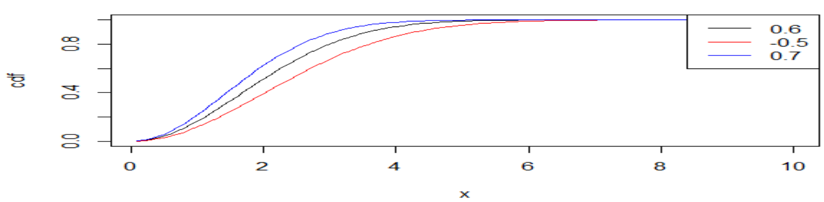
Definition 1.1. ***Rayleigh distribution***

The Rayleigh distribution is one of the most used distributions. The Rayleigh distribution was introduced by Rayleigh in 1880 and it has appeared as a special case of the Weibull distribution. It plays a key role in modeling and analyzing life-time data such as project effort loading modeling, survival and reliability analysis, theory of communication physical sciences, technology, diagnostic imaging, applied statistics and clinical research.
Let X the random variable follows Rayleigh distribution with scale parameter α, then its probability density function(pdf) and cumulative distribution function(cdf) take the form

$$f(x,\alpha) = \frac{x}{\alpha^2}\; e^{-\frac{x^2}{2\alpha^2}} \qquad x > 0 \qquad (3)$$



$$F(x,\alpha) = 1 - \; e^{-\frac{x^2}{2\alpha^2}} \qquad (4)$$

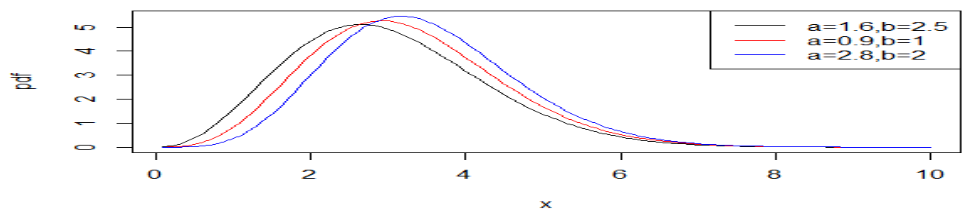


Definition 1.2: ***Exponentiated Rayleigh***

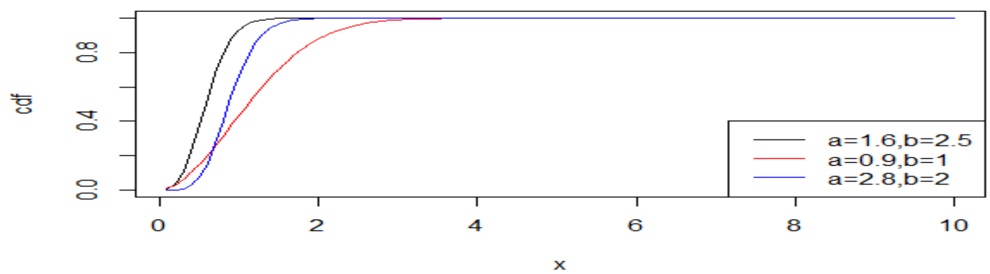
Let F(x, α) be the CDF of the Rayleigh distribution given by (1-3),we note that a is the parameter of GLM andα is the scale parameter of Rayleigh distribution . The CDF of Rayleigh Exponentiated can be defined by substituting (1-3) into equation (1-2). Hence the cdf and pdf of Rayleigh Exponentiated are

$$\mathrm{f(x)} = \mathrm{a}\frac{x}{\alpha^2}\; e^{-\frac{x^2}{2\alpha^2}}\left[1 - \; e^{-\frac{x^2}{2\alpha^2}}\right]^{a-1} \quad a > 0, x > 0 \qquad (5)$$

where a is the parameter of GLM and is α the scale parameter of Rayleigh distribution



$$F(x) = \left[1 - e^{\frac{x^2}{2\alpha^2}}\right]^a \quad a > 0, x > 0 \qquad (6)$$



2. Results and discussion

The main results are studying new subject does not study before and it is very important ,we studied mathematical properties for Exponentiated Rayleigh and we recommend it to researcher to study applied side of this properties.

2.1. Harmonic Mean for Exponentiated Rayleigh

The Harmonic mean of Exponentiated Rayleigh distribution is defined as

$$\frac{1}{H} = \int_0^\infty \frac{1}{x} f(x)dx \qquad (7)$$

Substitute into Equation No. (7) from (5) and integrate to get the Harmonic mean

$$\frac{1}{H} = \int_0^\infty \frac{1}{x} a \frac{x}{\alpha^2} \; e^{-\frac{x^2}{2\alpha^2}} \left[1 - \; e^{-\frac{x^2}{2\alpha^2}}\right]^{a-1} \quad dx$$

$$\frac{1}{H} = \int_0^\infty \frac{a}{\alpha^2} \; e^{-\frac{x^2}{2\alpha^2}} \left[1 - \; e^{-\frac{x^2}{2\alpha^2}}\right]^{a-1} dx$$

By using expanded binomial

$$\left[1 - e^{-\frac{x^2}{2\alpha^2}}\right]^{a-1} = \sum_{i=1}^{\infty} (-1)^k \binom{a-1}{k} \left[e^{-\frac{x^2}{2\alpha^2}}\right]^k$$

$$\frac{1}{H}=\frac{a}{\alpha^2}\int_0^{\infty}\sum_{i=1}^{\infty}(-1)^k\binom{a-1}{k}\left[e^{-\frac{x^2}{2\alpha^2}}\right]^{k+1}dx$$

$$\frac{1}{H}=\frac{a}{\alpha^2}\sum_{i=1}^{\infty}(-1)^k\binom{a-1}{k}\int_0^{\infty}\left[e^{-\frac{x^2}{2\alpha^2}}\right]^{k+1}dx \quad (8)$$

assuming that $u=\frac{(k+1)x^2}{2\alpha^2}$, we get $du=\frac{2(k+1)xdx}{2\alpha^2}$ to get value $dx=\frac{\alpha^2 du}{(k+1)x}=\frac{\alpha du}{(k+1)^{\frac{1}{2}}\sqrt{2}\sqrt{u}}$ we substituted into equation no(8) and simplified , we get

$$\frac{1}{H}=\frac{a}{\alpha^2}\sum_{i=1}^{\infty}(-1)^k\binom{a-1}{k}\int_0^{\infty}e^{-u}\frac{\alpha du}{(k+1)^{\frac{1}{2}}\sqrt{2}\sqrt{u}}$$

$$\frac{1}{H}=\frac{a}{(k+1)^{\frac{1}{2}}\sqrt{2}\alpha}\sum_{i=1}^{\infty}(-1)^k\binom{a-1}{k}\int_0^{\infty}e^{-u}\frac{du}{\sqrt{u}}$$

Applying the general definition of the gamma function

$$\Gamma(n)=\int_0^{\infty}x^{n-1}e^{-x}dx$$

$$\frac{1}{H}=\frac{a\sqrt{\pi}}{(k+1)^{\frac{1}{2}}\sqrt{2}\alpha}\sum_{k=1}^{\infty}(-1)^k\binom{a-1}{k}$$

2.2. Moment Generating Function for Exponentiated Rayleigh

The moment generating function is a function often used to characterize the distribution of a random variable. The moment generating function has great practical relevance because it can be used to easily derive moments; its derivatives at zero are equal to the moments of the random variable

Definition 1.3

Let x be a random variable. If E(x) exists and finites for all real number belongs to a closed interval [-h,h] , with h>0 then we say that X possesses a moment generating function and the function

$$E(e^{xt})=\int_0^{\infty}e^{xt}f(x)dx \qquad (9)$$

Substitute into Equation No. (9) from (5)and integrate to get the moment generating function

$$E(e^{xt})=\int_0^{\infty}e^{xt}a\frac{x}{\alpha^2}\,e^{-\frac{x^2}{2\alpha^2}}\left[1-e^{-\frac{x^2}{2\alpha^2}}\right]^{a-1}dx$$

Where a is parameter of GLM ,α is shape parameter of Rayleigh distribution

By using expanded binomial

$$\left[1-e^{-\frac{x^2}{2\alpha^2}}\right]^{a-1}=\sum_{i=1}^{\infty}(-1)^k\binom{a-1}{k}\left[e^{-\frac{x^2}{2\alpha^2}}\right]^k$$

$$E(e^{xt})=\sum_{i=1}^{\infty}\frac{a}{\alpha^2}(-1)^k\binom{a-1}{k}\int_0^{\infty}x\ e^{\frac{-(k+1)x^2+2\alpha^2tx}{2\alpha^2}}dx$$

$$E(e^{xt})=\sum_{i=1}^{\infty}\frac{a}{\alpha^2}(-1)^k\binom{a-1}{k}\int_0^{\infty}x\ e^{\frac{-(k+1)\left\{x^2-\frac{tx}{(k+1)}\right\}}{2\alpha^2}}dx$$

$$\frac{-(k+1)\left\{x^2-\frac{tx}{(k+1)}\right\}}{2\alpha^2}=\frac{-(k+1)}{2\alpha^2}\left\{\left(x-\frac{t}{2(k+1)}\right)^2-\frac{t^2}{4(k+1)^2}\right\}$$

$$\frac{-(k+1)\left\{x^2-\frac{tx}{(k+1)}\right\}}{2\alpha^2}=\left\{\frac{-(k+1)}{2\alpha^2}\left(x-\frac{t}{2(k+1)}\right)^2+\frac{t^2}{8\alpha^2(k+1)}\right\}$$

$$E(e^{xt})=\sum_{i=1}^{\infty}\frac{a}{\alpha^2}(-1)^k\binom{a-1}{k}\int_0^{\infty}x\ e^{\left\{\frac{-(k+1)}{2\alpha^2}\left(x-\frac{t}{2(k+1)}\right)^2+\frac{t^2}{8\alpha^2(k+1)}\right\}}dx$$

$$E(e^{xt})=\sum_{i=1}^{\infty}\frac{a}{\alpha^2}(-1)^k\binom{a-1}{k}e^{+\frac{t^2}{8\alpha^2(k+1)}}\int_0^{\infty}x\ e^{\left\{\frac{-(k+1)}{2\alpha^2}\left(x-\frac{t}{2(k+1)}\right)^2\right\}}dx \quad (10)$$

assuming that $u=\frac{(k+1)}{2\alpha^2}\left(x-\frac{t}{2(k+1)}\right)^2$ Differentiating, $du=\frac{(k+1)}{\alpha^2}\left(x-\frac{t}{2(k+1)}\right)dx$

to get value $dx=\frac{\alpha^2du}{(k+1)\left\{x-\frac{t}{2(k+1)}\right\}}$ Substituting in equation No (10). and simplifying, we get

$$E(e^{xt})=\sum_{i=1}^{\infty}\frac{a}{\alpha^2}(-1)^k\binom{a-1}{k}e^{+\frac{t^2}{8\alpha^2(k+1)}}\int_0^{\infty}\left\{\frac{\sqrt{u}\alpha\sqrt{2}}{\sqrt{k+1}}+\frac{t}{2(k+1)}\right\}e^{\{-u\}}\frac{\alpha du}{\sqrt{2}\sqrt{u}} \quad (11)$$

Let $K_1=\int_0^{\infty}\frac{\sqrt{u}\alpha\sqrt{2}}{\sqrt{k+1}}e^{\{-u\}}\frac{\alpha du}{\sqrt{2}\sqrt{u}}$ and $K_2=\int_0^{\infty}\frac{t}{2(k+1)}e^{\{-u\}}\frac{\alpha du}{\sqrt{2}\sqrt{u}}$

Applying the general definition of the gamma function

$$K_1=\int_0^{\infty}\frac{\sqrt{u}\alpha\sqrt{2}}{\sqrt{k+1}}e^{\{-u\}}\frac{\alpha du}{\sqrt{2}\sqrt{u}}$$

$$K_1=\frac{\alpha^2}{\sqrt{k+1}}\int_0^{\infty}u^0\,e^{\{-u\}}du=\frac{\alpha^2}{\sqrt{k+1}}\Gamma(1)$$

and $$K_2 = \frac{\alpha t}{2\sqrt{2}(k+1)} \int_0^\infty u^{-\frac{1}{2}} e^{\{-u\}} du = \frac{\alpha t}{2\sqrt{2}(k+1)} \Gamma(\frac{1}{2})$$

Applying the general definition of the gamma function

$$E(e^{xt}) = \sum_{i=1}^{\infty} \frac{a}{\alpha^2} (-1)^k \binom{a-1}{k} e^{+\frac{t^2}{8\alpha^2(k+1)}} \left[\frac{\alpha^2}{\sqrt{k+1}} + \frac{\alpha t}{2\sqrt{2}(k+1)} \Gamma(\frac{1}{2})\right]$$

2.3. Characteristic Function: for Exponentiated Rayleigh

Let X is a random variable the characteristic function of X is defined by

$$\emptyset_x = E(e^{ixt}) = \int_0^\infty e^{itx} f(x) dx \qquad (12)$$

Substitute into Equation No. (12) from $(m(t)_n = \frac{1}{R(T)} \int_t^\infty (x-t)^n f(x) dx)$ and integrate to get the characteristic function

$$E(e^{ixt}) = \int_0^\infty e^{xit} a \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}} \left[1 - e^{-\frac{x^2}{2\alpha^2}}\right]^{a-1} dx$$

By using expanded binomial

$$\left[1 - e^{-\frac{x^2}{2\alpha^2}}\right]^{a-1} = \sum_{i=1}^{\infty} (-1)^k \binom{a-1}{k} \left[e^{-\frac{x^2}{2\alpha^2}}\right]^k$$

$$E(e^{ixt}) = \sum_{i=1}^{\infty} \frac{a}{\alpha^2} (-1)^k \binom{a-1}{k} \int_0^\infty x\ e^{\frac{-(k+1)x^2 + 2\alpha^2 tix}{2\alpha^2}} dx$$

$$E(e^{xt}) = \sum_{i=1}^{\infty} \frac{a}{\alpha^2} (-1)^k \binom{a-1}{k} \int_0^\infty x\ e^{\frac{-(k+1)\left\{x^2 - \frac{txi}{(k+1)}\right\}}{2\alpha^2}} dx$$

$$\frac{-(k+1)\left\{x^2 - \frac{tix}{(k+1)}\right\}}{2\alpha^2} = \frac{-(k+1)}{2\alpha^2} \left\{\left(x - \frac{ti}{2(k+1)}\right)^2 + \frac{t^2}{4(k+1)^2}\right\}$$

$$\frac{-(k+1)\left\{x^2 - \frac{tix}{(k+1)}\right\}}{2\alpha^2} = \left\{\frac{-(k+1)}{2\alpha^2}\left(x - \frac{it}{2(k+1)}\right)^2 - \frac{t^2}{8\alpha^2(k+1)}\right\}$$

$$E(e^{xt}) = \sum_{i=1}^{\infty} \frac{a}{\alpha^2} (-1)^k \binom{a-1}{k} \int_0^\infty x\ e^{\left\{\frac{-(k+1)}{2\alpha^2}\left(x - \frac{it}{2(k+1)}\right)^2 - \frac{t^2}{8\alpha^2(k+1)}\right\}} dx$$

$$E(e^{xt}) = \sum_{i=1}^{\infty} \frac{a}{\alpha^2} (-1)^k \binom{a-1}{k} e^{-\frac{t^2}{8\alpha^2(k+1)}} \int_0^\infty x\ e^{\left\{\frac{-(k+1)}{2\alpha^2}\left(x - \frac{it}{2(k+1)}\right)^2\right\}} dx \qquad (13)$$

assuming that $u = \frac{(k+1)}{2\alpha^2}\left(x - \frac{ti}{2(k+1)}\right)^2$ Differentiating, $du = \frac{(k+1)}{\alpha^2}\left(x - \frac{it}{2(k+1)}\right) dx$

to get value $dx = \frac{\alpha^2 du}{(k+1)\left\{x-\frac{it}{2(k+1)}\right\}}$ Substituting in equation No (13). and simplifying, we get

$$E(e^{xt}) = \sum_{i=1}^{\infty} \frac{a}{\alpha^2} (-1)^k \binom{a-1}{k} e^{-\frac{t^2}{8\alpha^2(k+1)}} \int_0^{\infty} \left\{ \frac{\sqrt{u}\alpha\sqrt{2}}{\sqrt{k+1}} + \frac{t}{2(k+1)} \right\} e^{\{-u\}} \frac{\alpha du}{\sqrt{2}\sqrt{u}} \quad (14)$$

Let $K_1 = \int_0^{\infty} \frac{\sqrt{u}\alpha\sqrt{2}}{\sqrt{k+1}} e^{\{-u\}} \frac{\alpha du}{\sqrt{2}\sqrt{u}}$ and $K_2 = \int_0^{\infty} \frac{t}{2(k+1)} e^{\{-u\}} \frac{\alpha du}{\sqrt{2}\sqrt{u}}$

Applying the general definition of the gamma function

$$K_1 = \int_0^{\infty} \frac{\sqrt{u}\alpha\sqrt{2}}{\sqrt{k+1}} e^{\{-u\}} \frac{\alpha du}{\sqrt{2}\sqrt{u}}$$

$$K_1 = \frac{\alpha^2}{\sqrt{k+1}} \int_0^{\infty} u^0 \, e^{\{-u\}} du = \frac{\alpha^2}{\sqrt{k+1}} \Gamma(1)$$

and $$K_2 = \frac{\alpha t}{2\sqrt{2}(k+1)} \int_0^{\infty} u^{-\frac{1}{2}} e^{\{-u\}} du = \frac{\alpha t}{2\sqrt{2}(k+1)} \Gamma(\frac{1}{2})$$

Applying the general definition of the gamma function

$$E(e^{xt}) = \sum_{i=1}^{\infty} \frac{a}{\alpha^2} (-1)^k \binom{a-1}{k} e^{-\frac{t^2}{8\alpha^2(k+1)}} \left[\frac{\alpha^2}{\sqrt{k+1}} + \frac{\alpha t}{2\sqrt{2}(k+1)} \Gamma(\frac{1}{2}) \right]$$

2.4. Quantile Function for Exponentiated Rayleigh

Let X denotes the random variable with pdf given in ($m(t)_n = \frac{1}{R(T)} \int_t^{\infty} (x - t)^n \, f(x) dx$). The quintile function, say $Q(u)$ is defined as $F[Q(u)]$

$$\text{U} = \left[1 - e^{\frac{x^2}{2\alpha^2}} \right]^a \quad a > 0, x > \qquad (15)$$

Here u is the uniform random variable defined on the unit interval (0,1).

$$e^{-\frac{x^2}{2\alpha^2}} = 1 - \; u^{\frac{1}{a}}$$

$$-\frac{x^2}{2\alpha^2} = lin(1 - \; u^{\frac{1}{a}})$$

$$x^2 = -2\alpha^2 lin(1 - \; u^{\frac{1}{a}})$$

$$Q(u) = \sqrt{2\alpha^2 lin \frac{1}{(1 - \; u^{\frac{1}{a}})}}$$

The limitations of classical measure of skewness and kurtosis are well-known when the moments don't exist for any distribution. To overcome these shortcomings the analysis of variability of the skewness and kurtosis can be investigated based on quintile measures. The Bowley measure of skewness based on quintiles is given as

$$B = \frac{Q(0.75) - 2Q(0.5) + Q(0.25)}{Q(0.75) - Q(0.25)}$$

Similarly, the Moors measure of kurtosis based on octiles is defined by

$$M = \frac{Q(0.875) - Q(.125) + Q(0.375) - Q(0.625)}{Q(0.75) - Q(0.25)}$$

2.5. Raw Moments: For Exponentiated Rayleigh

The rth moment about origin μr′ (raw moment) is generally defined as

$$\acute{\mu}_r = \int_0^\infty x^r f(x)dx \qquad (16)$$

Substitute into Equation No. (2-11) from ($\boldsymbol{\gamma_4} = \frac{\acute{\mu}_4 + 6\acute{\mu}_1{}^2\acute{\mu}_2 - 4\acute{\mu}_1\acute{\mu}_3 - 3\acute{\mu}_1{}^4}{[\acute{\mu}_2 - (\acute{\mu}_1)^2]^2}$) and integrate to get the raw moment

$$\acute{\mu}_r = \int_0^\infty \mathrm{a}\frac{\mathrm{x}^{r+1}}{\alpha^2} \ \mathrm{e}^{-\frac{\mathrm{x}^2}{2\alpha^2}}\left[1 - \ \mathrm{e}^{-\frac{\mathrm{x}^2}{2\alpha^2}}\right]^{\mathrm{a}-1} dx$$

By using expanded binomial

$$\left[1 - \mathrm{e}^{-\frac{\mathrm{x}^2}{2\alpha^2}}\right]^{\mathrm{a}-1} = \sum_{\mathrm{i}=1}^{\infty}(-1)^{\mathrm{k}}\binom{\mathrm{a}-1}{\mathrm{k}}\left[\mathrm{e}^{-\frac{\mathrm{x}^2}{2\alpha^2}}\right]^{\mathrm{k}}$$

$$\acute{\mu}_r = \sum_{\mathrm{l}=1}^{\infty}(-1)^{\mathrm{k}}\binom{\mathrm{a}-1}{\mathrm{k}}\int_0^\infty \mathrm{a}\frac{\mathrm{x}^{r+1}}{\alpha^2} \ \mathrm{e}^{-\frac{(\mathrm{k}+1)\mathrm{x}^2}{2\alpha^2}}dx \qquad (17)$$

assuming that $u = \frac{(k+1)x^2}{2\alpha^2}$Differentiating, we get $du = \frac{(k+1)2xdx}{2\alpha^2}$ to get value $dx = \frac{\alpha^2 du}{x(k+1)} = \frac{\boldsymbol{\alpha} du}{(k+1)^{\frac{-1}{2}}\sqrt{2}\sqrt{u}}$ by Substituting in equation No (17). and simplifying , we get

$$\acute{\mu}_r = \frac{\mathrm{a}}{\alpha^2}\sum_{\mathrm{l}=1}^{\infty}(-1)^{\mathrm{k}}\binom{\mathrm{a}-1}{\mathrm{k}}\int_0^\infty \frac{2^{\frac{r+1}{2}}\alpha^{r+1}u^{\frac{r+1}{2}}}{(k+1)^{\frac{r+1}{2}}} \ \mathrm{e}^{-\mathrm{u}}\frac{\boldsymbol{\alpha} du}{(k+1)^{\frac{1}{2}}\sqrt{2}\sqrt{u}}$$

$$\acute{\mu}_r = \frac{a\alpha^r 2^{\frac{r}{2}}}{(k+1)^{\frac{r+2}{2}}}\sum_{\mathrm{l}=1}^{\infty}(-1)^{\mathrm{k}}\binom{\mathrm{a}-1}{\mathrm{k}}\int_0^\infty u^{\frac{r}{2}} \ \mathrm{e}^{-\mathrm{u}}du$$

Applying the general definition of the gamma function

$$\acute{\mu}_r = \frac{a\alpha^r 2^{\frac{r}{2}}}{(k+1)^{\frac{r+2}{2}}} \sum_{i=1}^{\infty} (-1)^k \binom{a-1}{k} \; \Gamma(\frac{r}{2}+1)$$

2.6. Moments of Residual Life For Exponentiated Rayleigh (MRL):

The nth moments of residual life is denoted by $E[(x-t)^n | \, x > t]$ $where \;\; n = 1,2,3, \dots ..$

Is defined by $m(t)_n = \frac{1}{R(T)} \int_t^{\infty} (x-t)^n \, f(x) dx$ (18)

Substitute in the general form for the residuals life for the probability density function of the Rayleigh distribution we get

$$m(t)_n = \frac{1}{R(T)} \int_t^{\infty} (x-t)^n e^{-\frac{x^2}{2\alpha^2}} \left[1 - \;\; e^{-\frac{x^2}{2\alpha^2}}\right]^{a-1} \; dx$$

By using expanded binomial

$$\left[1 - e^{-\frac{x^2}{2\alpha^2}}\right]^{a-1} = \sum_{i=1}^{\infty} (-1)^k \binom{a-1}{k} \left[e^{-\frac{x^2}{2\alpha^2}}\right]^k$$

But from the binomial theorem, we get

$$(x-t)^n = (-t)^n (1 - \frac{x}{t})^n = (-t)^n \sum_{h=0}^{\infty} (-1)^h \binom{n}{h} (\frac{x}{t})^h$$

Using the binomial expansion of the expression ($(x-t)^n$) and the substitution of the general form for residual life

$$m(t)_n = \frac{\sum_{h=0}^{\infty} (-1)^h \binom{n}{h} \sum_{i=1}^{\infty} (-1)^k \binom{a-1}{k}}{R(T)} \int_t^{\infty} (-t)^n (\frac{x}{t})^h \frac{x}{\alpha^2} \; e^{-\frac{(1+k)x^2}{2\alpha^2}} \; dx$$

$$m(t)_n = (-t)^n \frac{\sum_{h=0}^{\infty} (-1)^h \binom{n}{h} \sum_{i=1}^{\infty} (-1)^k \binom{a-1}{k}}{R(T) t^h} \int_t^{\infty} x^{h+1} \;\; e^{-\frac{(1+k)x^2}{2\alpha^2}} dx \qquad (19)$$

assuming that $u = \frac{(1+k)x^2}{2\alpha^2}$Differentiating, we get $du = \frac{x(k+1)dx}{\alpha^2}$ to get value $dx = \frac{\alpha^2 du}{x} = \frac{\boldsymbol{\alpha} du}{(k+1)^{\frac{-1}{2}} \sqrt{2}\sqrt{u}}$ Substituting in equation No (19). and simplifying, we get

$$\mathrm{m(t)_n} = 2^{\frac{h}{2}} \frac{\alpha^{h+2} (-t)^n \sum_{h=0}^{\infty} (-1)^h \binom{n}{h} 1}{t^h \alpha^2 R(T) (1+k)^{\frac{h}{2}}} \int_t^{\infty} \;\; u^{\frac{h}{2}} e^{-u} \frac{du}{}$$

$$m(t)_n = 2^{\frac{h}{2}} \frac{\alpha^{h+2}(-t)^n \sum_{h=0}^{\infty}(-1)^h \binom{n}{h} 1}{t^h \alpha^2 R(T)(1+k)^{\frac{h}{2}}} \int_t^{\infty} u^{\frac{h}{2}} \; e^{-u} du$$

using the general form of the gamma function $\boldsymbol{\Gamma}(a,b) = \int_b^{\infty} y^{a-1} e^{-y}\, dy$

$$m(t)_n = 2^{\frac{h}{2}} \frac{\alpha^{h+2}(-t)^n \sum_{h=0}^{\infty}(-1)^h \binom{n}{h} 1}{t^h \alpha^2 R(T)(1+k)^{\frac{h}{2}}} \boldsymbol{\Gamma}\left(\frac{h+1}{2}, t\right)$$

***2.7. Mean of Residual life for Exponentiated Rayleigh*:**

The mean residual of (Rayleigh) distribution is defined by

$$m(t) = \frac{1}{R(T)} \int_t^{\infty} a \frac{x^2}{\alpha^2} \; e^{-\frac{x^2}{2\alpha^2}} \left[1 - \; e^{-\frac{x^2}{2\alpha^2}}\right]^{a-1} dx - t \quad (20)$$

By using expanded binomial

$$\left[1 - e^{-\frac{x^2}{2\alpha^2}}\right]^{a-1} = \sum_{i=1}^{\infty} (-1)^k \binom{a-1}{k} \left[e^{-\frac{x^2}{2\alpha^2}}\right]^k$$

$$m(t) = \frac{\sum_{i=1}^{\infty}(-1)^k \binom{a-1}{k}}{R(T)} \int_t^{\infty} a \frac{x^2}{\alpha^2} \; e^{-\frac{(k+1)x^2}{2\alpha^2}} dx - t \qquad (21)$$

assuming that $u = \frac{(k+1)x^2}{2\alpha^2}$Differentiating, we get $du = \frac{2x(k+1)dx}{2\alpha^2}$ to get value $dx = \frac{\alpha^2 du}{x(k+1)} = \frac{\boldsymbol{\alpha} du}{(k+1)^{\frac{-1}{2}} \sqrt{2}\sqrt{u}}$ Substituting in equation No (21). and simplifying, we get

$$m(t) = \sqrt{2} \frac{\boldsymbol{\alpha} a \sum_{i=1}^{\infty}(-1)^k \binom{a-1}{k}}{R\alpha(T)(k+1)^{\frac{1}{2}}} \int_t^{\infty} u^{\frac{1}{2}} \; e^{-u} du - t$$

using the general form of the gamma function $\boldsymbol{\Gamma}(a,b) = \int_b^{\infty} y^{a-1} e^{-y}\, dy$

$$m(t) = \sqrt{2} \frac{\boldsymbol{\alpha} a \sum_{i=1}^{\infty}(-1)^k \binom{a-1}{k}}{R\alpha(T)(k+1)^{\frac{1}{2}}} \int_t^{\infty} u^{\frac{1}{2}} \; e^{-u} du - t$$

$$m(t) = \sqrt{2} \frac{\boldsymbol{\alpha} a \sum_{i=1}^{\infty}(-1)^k \binom{a-1}{k}}{R\alpha(T)(k+1)^{\frac{1}{2}}} \boldsymbol{\Gamma}\left(\frac{3}{2}, t\right) - t$$

2.8 Reversed Residual Life For Exponentiated Rayleigh

The nth moments of residual life denoted by $E[(t-x)^n |\, x \le t]$ *where* $n = 1,2,3, \dots ..$

Is defined by $M(t)_n = \frac{1}{F(T)} \int_0^t (t-x)^n \, f(x) dx$ (22)

Substitute in the general form for the residuals life for the probability density function of the Rayleigh distribution the reversed residual life becomes

$$M(t)_n = \frac{1}{F(T)} \int_0^t (t-x)^n \, \mathrm{a} \frac{\mathrm{x}}{\alpha^2} \ \mathrm{e}^{-\frac{\mathrm{x}^2}{2\alpha^2}} \left[1 - \ \mathrm{e}^{-\frac{\mathrm{x}^2}{2\alpha^2}}\right]^{\mathrm{a}-1} dx$$

Using the binomial expansion of the expression

$$(t-x)^n = (t)^n (1 - \frac{x}{t})^n = (t)^n \sum_{\omega=0}^{\infty} (-1)^{\omega} \binom{n}{\omega} (\frac{x}{t})^{\omega}$$

By using expanded binomial

$$\left[1 - \mathrm{e}^{-\frac{\mathrm{x}^2}{2\alpha^2}}\right]^{\mathrm{a}-1} = \sum_{\mathrm{i}=1}^{\infty} (-1)^{\mathrm{k}} \binom{\mathrm{a}-1}{\mathrm{k}} \left[\mathrm{e}^{-\frac{\mathrm{x}^2}{2\alpha^2}}\right]^{\mathrm{k}}$$

$$M(t)_n = \frac{(\mathrm{t})^{\mathrm{n}} \sum_{\omega=0}^{\infty} (-1)^{\omega} \binom{\mathrm{n}}{\omega} (\frac{\mathrm{x}}{\mathrm{t}})^{\omega} \sum_{\mathrm{i}=1}^{\infty} (-1)^{\mathrm{k}} \binom{\mathrm{a}-1}{\mathrm{k}}}{\mathrm{F(T)}} \int_0^{\mathrm{t}} \mathrm{a} \frac{\mathrm{x}}{\alpha^2} \ \mathrm{e}^{-\frac{(\mathrm{k}+1)\mathrm{x}^2}{2\alpha^2}} \mathrm{dx} \quad (23)$$

Where, a is the parameter of GLM ,α is shape parameter of Rayleigh distribution

assuming that $u = \frac{(k+1)x^2}{2\alpha^2}$Differentiating, we get $du = \frac{2x(k+1)dx}{2\alpha^2}$ to get value

$dx = \frac{\alpha^2 du}{x(k+1)} = \frac{\boldsymbol{\alpha} du}{(k+1)^{\frac{-1}{2}} \sqrt{2}\sqrt{u}}$ Substituting in equation No (23). and simplifying, we get

$$M(t)_n = \frac{(\mathrm{t})^{\mathrm{n}} \sum_{\omega=0}^{\infty} (-1)^{\omega} \binom{\mathrm{n}}{\omega} (\frac{\mathrm{X}}{\mathrm{t}})^{\omega} \sum_{\mathrm{i}=1}^{\infty} (-1)^{\mathrm{k}} \binom{\mathrm{a}-1}{\mathrm{k}}}{\mathrm{F(T)}} \int_0^t \ e^{-u} \quad du$$

Applying the general definition of the gamma function

$$M(t)_n = \frac{(t)^n \sum_{\omega=0}^{\infty} (-1)^{\omega} \binom{n}{\omega} [\alpha\sqrt{2}]^{w+1}}{\alpha F(T) t^w \sqrt{2}} \Gamma(+1)$$

3. Conclusion

In this research we introduced the Rayleigh distribution and study some new mathematical properties for Exponentiated Rayleigh, including Harmonic Mean, Moment Generating Function, Characteristic Function, Quantile function, Raw Moments, Moments of Residual life (MRL), Reversed Residual Life, Mean of Residual life. Hence, we invite researchers to study more mathematical properties of distributions because of its many applications that can help solve many life problems.

CONFLICT OF INTEREST

There is no conflict of interest.

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