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Continuous K-g-fusion frames in Hilbert C^*-modules

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Abstract.

In this paper, we introduce the concept of continuous g-fusion frame and K-g-fusion frame in Hilbert C^*-modules. Furthermore, we investigate some properties of them and discuss the perturbation problem for continuous K-g-fusion frames.

Keywords: Continuous fusion frame; Continuous g-fusion frame; Continuous K-g-fusion frame; C^*-algebra; Hilbert C^*-module.

1. Introduction and preliminaries

Frame theory has a great revolution in recent years it was introduced by Duffin and Schaeffer [7] in 1952 to study some deep problems in nonharmonic Fourier series. In 2000, Frank and Larson [9] introduced the concept of frames in Hilbert C^*-modules as a generalization of frames in Hilbert spaces. The basic idea was to consider modules over C^*-algebras of linear spaces and to allow the inner product to take values in C^*-algebras [15]. A. Khosravi and B. Khosravi [14] introduced the fusion frames and $g-$frames in Hilbert C^*-modules. Afterwards, A. Alijani and M. Dehghan consider frames with

C^*-valued bounds [2] in Hilbert C^*-modules. Recently, Fakhr-dine Nhari et al. [18] introduced the concepts of g-fusion frame and K-g-fusion frame in Hilbert C^*-modules. Many generalizations of the concept of frame have been defined in Hilbert C^*-modules (see [10, 20, 21, 22, 23, 24, 25] for more detail).

The notion of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by Kaiser [11] and independently by Ali, Antoine and Gazeau [1], and these frames are known as continuous frames.

The paper is organized as follows: We continue this introductory section we briefly recall the definitions and basic properties of C^*-algebra and Hilbert C^*-modules. In Section 2, we introduce the concept of continuous g-fusion frame, the continuous g-fusion frame operator and establish some results. In Section 3, we introduce the concept of continuous K-g-fusion frame and gives some properties. Finally, in Section 4, we discuss the perturbation problem for continuous K-g-fusion frame.

In the following, we briefly recall the definitions and basic properties of C^*-algebra, Hilbert C^*-modules. The references for C^*-algebras are [4, 6]. For a C^*-algebra $\mathcal{A}$ if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 0.1. [4]. If $\mathcal{A}$ is a Banach algebra, then an involution is a map $a \mapsto a^*$ of $\mathcal{A}$ into itself such that for all a and b in $\mathcal{A}$ and all scalars α the following conditions hold:

(1) $(a^*)^* = a$;

(2) $(ab)^* = b^* a^*$;

(3) $(\alpha a + b)^* = \bar{\alpha} a^* + b^*$.

Definition 0.2. [4]. A C^*-algebra $\mathcal{A}$ is a Banach algebra with involution such that

$$\|a^* a\| = \|a\|^2$$

for all a in $\mathcal{A}$.

Example 0.3. Let $\mathcal{B} = B(\mathcal{H})$ be the algebra of bounded operators on a Hilbert space $\mathcal{H}$. Then $\mathcal{B} = B(\mathcal{H})$ is a C^*-algebra, where, for each operator A, A^* is the adjoint of A.

Definition 0.4. [12]. Let $\mathcal{A}$ be a unital C^*-algebra and U be a left $\mathcal{A}$-module such that the linear structures of $\mathcal{A}$ and U are compatible. Then U is a pre-Hilbert $\mathcal{A}$-module if

U is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : U \times U \to \mathcal{A}$ which is sesquilinear and positive definite. In the other words,

(i) $\langle x, x \rangle \geq 0$ for all $x \in U$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(ii) $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in U$;

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in U$.

For $x \in U$, we define $||x|| = ||\langle x, x \rangle||^{\frac{1}{2}}$. If U is complete with $|| \cdot ||$, then it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $\|a\| = (a^*a)^{\frac{1}{2}}$.

Throughout this paper, U is considered to be a Hilbert C^*-module over a C^*-algebra, and we denote that I_U is a countable index set, $\{H_w\}_{w\in\Omega}$ is a sequence of Hilbert C^*-submodules over U and $\{V_w\}_{w\in\Omega}$ is a sequence of Hilbert C^*-modules.

We denote that $End^*_{\mathcal{A}}(U, V_w)$ is a set of all adjointable operators. In particular $End^*_{\mathcal{A}}(U)$ denote the set of all bounded linear operators on U. We denote $\mathcal{R}(T)$ for the range of T.

The following lemmas will be used to prove our mains results.

Lemma 0.5. [19]. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. If $T \in End^*_{\mathcal{A}}(\mathcal{H})$, then*

$$\langle Tx, Tx \rangle \leq \|T\|^2 \langle x, x \rangle, \forall x \in \mathcal{H}.$$

Lemma 0.6. [3]. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:*

(i) *T is surjective.*

(ii) *T^* is bounded below with respect to norm, i.e., there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$ for all $x \in \mathcal{K}$.*

(iii) *T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $\langle T^*x, T^*x \rangle \geq m' \langle x, x \rangle$ for all $x \in \mathcal{K}$.*

Lemma 0.7. [2]. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$.*

(i) *If T is injective and T has a closed range, then the adjointable map T^*T is invertible and*

$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2.$$

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

Lemma 0.8. [3] *Let H be a Hilbert $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$, and $T \in End^*_{\mathcal{A}}(H)$ such that $T^* = T$. The following statements are equivalent:*

(i) *T is surjective.*

(ii) *There are $m, M > 0$ such that $m\|x\| \leq \|Tx\| \leq M\|x\|$ for all $x \in H$.*

(iii) *There are $m', M' > 0$ such that $m'\langle x, x\rangle \leq \langle Tx, Tx\rangle \leq M'\langle x, x\rangle$ for all $x \in H$.*

Lemma 0.9. [8] *Let $\mathcal{A}$ be a C^*-algebra and E, H and K be Hilbert $\mathcal{A}$-modules. Let $T \in End^*_{\mathcal{A}}(E, K)$ and $T^{'} \in End^*_{\mathcal{A}}(H, K)$ be such that $\overline{\mathcal{R}(T^*)}$ is orthogonally complemented. Then the following statements are equivalent:*

(1) *$T^{'}(T^{'})^* \leq \mu TT^*$ for some $\mu > 0$.*

(2) *There exists $\mu > 0$ such that $||(T^{'})^*z|| \leq \mu||T^*z||$ for all $z \in K$.*

(3) *There exists a solution $X \in End^*_{\mathcal{A}}(H, E)$ of the so-called Douglas equation $T^{'} = TX$.*

(3) *$\mathcal{R}(T^{'}) \subseteq \mathcal{R}(T)$.*

Lemma 0.10. [2] *If $\phi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism between C^*-algebras, then ϕ is positive and increasing, that is, $\phi(\mathcal{A}^+) \subseteq \mathcal{B}^+$, and if $a \leq b$, then $\phi(a) \leq \phi(b)$.*

2. Continuous g-fusion frames in Hilbert C^*-modules

Definition 0.11. Let $\{H_w\}_{w\in\Omega}$ be a sequence of closed submodules orthogonally complemented in U, P_{H_w} be the orthogonal projection from U to H_w, $\Lambda_w \in End^*_{\mathcal{A}}(U, V_w)$, $w \in \Omega$ and $\{v_w\}_{w\in\Omega}$ be a family of weights in $\mathcal{A}$, i.e., each v_w is a positive invertible element from the center of $\mathcal{A}$. We say $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a continuous g-fusion frame for U if

(1) for each $x \in U$, $\{P_{H_w}x\}_{w\in\Omega}$ is measurable;

(2) for each $x \in U$, the fonction $\tilde{\Lambda} : \Omega \to V_w$ defined by $\tilde{\Lambda}(w) = \Lambda_w x$ is measurable;

(3) there exist $0 < A \leq B < \infty$ such that

$$A\langle x, x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w) \leq B\langle x, x\rangle, \qquad \forall x \in U. \tag{0.1}$$

We call A and B the lower and upper continuous g-fusion frame bounds, respectively. If the right-hand inequality of (0.1) is satisfied, then we call Λ a continuous g-fusion Bessel sequence. If $A = B$, then we call Λ the tight continuous g-fusion frame. Moreover, If $A = B = 1$, then Λ is called the Parseval continuous g-fusion frame.

See [5, 13, 16, 17, 26] for more information on fusion frames and properties.

Proposition 0.12. *Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$. If Λ is a continuous g-fusion frame for U, then $\{v_w\Lambda_w P_{H_w}\}_{w\in\Omega}$ is a continuous g-frame for U.*

Proof. Since Λ is a continuous g-fusion frame, for each $x \in U$, we have

$$A\langle x, x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) \leq B\langle x, x\rangle.$$

Thus

$$A\langle x, x\rangle \leq \int_\Omega \langle v_w\Lambda_w P_{H_w} x, v_w\Lambda_w P_{H_w} x\rangle d\mu(w) \leq B\langle x, x\rangle.$$

Hence $\{v_w\Lambda_w P_{H_w}\}_{w\in\Omega}$ is a continuous g-fusion frame for U. □

Proposition 0.13. *If Λ is a continuous g-fusion Bessel sequence for U, then the operator $T_\Lambda : \oplus_{w\in\Omega}V_w \to U$, defined by $T_\Lambda(\{x_w\}_{w\in\Omega}) = \int_\Omega v_w P_{H_w}\Lambda_w^* x_w d\mu(w)$, for all $\{x_w\}_{w\in\Omega} \in \oplus_{w\in\Omega}V_w$, is adjointable bounded.*

Proof. Let $\{x_w\}_{w\in\Omega} \in \oplus_{w\in\Omega}V_w$. Then

$$\begin{aligned}
T_\Lambda(\{x_w\}_{w\in\Omega}) &= \sup_{||y||=1} ||\langle \int_\Omega v_w P_{H_w}\Lambda_w^* x_w d\mu(w), y\rangle|| \\
&= \sup_{||y||=1} ||\int_\Omega \langle x_w, v_w\Lambda_w P_{H_w} y\rangle d\mu(w)|| \\
&\leq \sup_{||y||=1} ||\int_\Omega \langle x_w, x_w\rangle d\mu(w)||^{\frac{1}{2}} ||\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} y, \Lambda_w P_{H_w} y\rangle d\mu(w)||^{\frac{1}{2}} \\
&\leq \sqrt{B}||\{x_w\}_{w\in\Omega}||.
\end{aligned}$$

So T_Λ is bounded. And we have, for all $y \in U$ and $\{x_w\}_{w\in\Omega} \in \oplus_{w\in\Omega}V_w$,

$$\begin{aligned}
\langle T_\Lambda\{x_w\}_{w\in\Omega}, y\rangle = \langle \int_\Omega v_w P_{H_w}\Lambda_w^* x_w d\mu(w), y\rangle &= \int_\Omega \langle x_w, v_w\Lambda_w P_{H_w} y\rangle d\mu(w) \\
&= \langle \{x_w\}_{w\in\Omega}, \{v_w\Lambda_w P_{H_w} y\}_{w\in\Omega}\rangle.
\end{aligned}$$

Thus $T_\Lambda^*(x) = \{v_w\Lambda_w P_{H_w} x\}_{w\in\Omega}$. □

Note that T_Λ is called the synthesis operator of Λ, and T_Λ^* is called the analysis operator of Λ.

Theorem 0.14. *If $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a continuous g-fusion frame for U with frame bounds A and B, then T_Λ is surjective with $||T_\Lambda|| \leq \sqrt{B}$ and T_Λ^* is injective and a closed range.*

Proof. We have, for all $x \in U$,

$$A\langle x, x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) \leq B\langle x, x\rangle.$$

Thus

$$A\langle x, x\rangle \leq \langle T_\Lambda^* x, T_\Lambda^* x\rangle \leq B\langle x, x\rangle. \tag{0.2}$$

Hence

$$\sqrt{A}||x|| \leq ||T_\Lambda^* x||. \tag{0.3}$$

So T_Λ^* is injective.

Now we will show that the $\mathcal{R}(T_\Lambda^*)$ is closed.

Let $\{T_\Lambda^*(x_n)\}_{n\in\mathbb{N}} \in \mathcal{R}(T_\Lambda^*)$ such that $\lim_n T_\Lambda^*(x_n) = y$. For n, $m \in \mathbb{N}$, we have, from (0.2),

$$\langle x_n - x_m, x_n - x_m\rangle \leq A^{-1}\langle T_\Lambda^*(x_n - x_m), T_\Lambda^*(x_n - x_m)\rangle.$$

Thus

$$||\langle x_n - x_m, x_n - x_m\rangle|| \leq A^{-1}||T_\Lambda^*(x_n - x_m)||^2.$$

Since $\{T_\Lambda^*(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\oplus_{w\in\Omega} V_w$, $||\langle x_n - x_m, x_n - x_m\rangle|| \to 0$. Therefore, the sequence $\{x_n\}_{n\mathbb{N}}$ is a Cauchy sequence in U and so there exists $x \in U$ such that $\lim_n x_n = x$. Again by (0.2), we have

$$||T_\Lambda^* x_n - T_\Lambda^* x||^2 \leq B||\langle x_n - x, x_n - x\rangle||$$

and thus $||T_\Lambda^*(x_n) - T_\Lambda^*(x)|| \to 0$ implies that $T_\Lambda^*(x) = y$ and hence $\mathcal{R}(T_\Lambda^*)$ is closed. Finally from (0.3) and Lemma 0.6, T_Λ is surjective. □

Definition 0.15. Let Λ be a continuous g-fusion frame for U. Define a continuous g-fusion frame operator S_Λ on U by $S_\Lambda x = T_\Lambda T_\Lambda^* x = \int_\Omega v_w^2 P_{H_w} \Lambda_w^* \Lambda_w P_{H_w} x d\mu(w)$ for all $x \in U$.

Theorem 0.16. *The continuous g-fusion frame operator S_Λ of Λ is bounded, positive, selfadjoint and invertible. Moreover,*

$$AI_U \leq S_\Lambda \leq BI_U. \tag{0.4}$$

Proof. It easy to see that the operator S_Λ is positive, selfadjoint and bounded.

Now from Theorem 0.14, T_Λ is surjective and by Lemma 0.7, $T_\Lambda T_\Lambda^*$ is invertible and so S_Λ is invertible.

We have, for all $x \in U$,

$$\langle S_\Lambda x, x\rangle = \langle \int_\Omega v_w^2 P_{H_w} \Lambda_w^* \Lambda_w P_{H_w} x d\mu(w), x\rangle = \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w).$$

From (0.1),

$$AI_U \leq S_\Lambda \leq BI_U.$$

This completes the proof. □

In this theorem, we give an equivalent definition of continuous g-fusion frame in Hilbert C^*-module.

Theorem 0.17. *Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$. Then Λ is a countinuous g-fusion frame for U if and only if there exist constants $0 < A \leq B < \infty$ such that*

$$A||x||^2 \leq || \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w)|| \leq B||x||^2, \qquad \forall x \in U. \tag{0.5}$$

Proof. If Λ is a continuous g-fusion frame for U, then we have the inequality (0.5).

Conversely, assume that (0.5) holds. Then the operator $T_\Lambda : \oplus_{w\in\Omega} V_w \to U$, defined by $T_\Lambda(\{x_w\}_{w\in\Omega}) = \int_\Omega v_w P_{H_w} \Lambda_w^* x_w d\mu(w)$, is well-defined, adjointable and bounded with $T_\Lambda^* : U \to \oplus_{w\in\Omega} V_w$ defined by $T_\Lambda^*(\{x_w\}_{w\in\Omega}) = \{v_w \Lambda_w P_{H_w} x\}_{w\in\Omega}$, and hence the operator $S_\Lambda = T_\Lambda T_\Lambda^*$ is selfadjoint and positive. From (0.5), we have, for all $x \in U$,

$$A||x||^2 \leq ||\langle T_\Lambda^* x, T_\Lambda^* x\rangle|| \leq B||x||^2$$

and so

$$\sqrt{A}||x|| \leq ||T_\Lambda^* x|| \leq \sqrt{B}||x||.$$

Hence

$$\sqrt{A}||x|| \leq ||\langle S_\Lambda^{\frac{1}{2}} x, S_\Lambda^{\frac{1}{2}} x\rangle|| \leq \sqrt{B}||x||$$

and so by Lemma 0.8, there exist two positive constants A' and B' such that

$$A'\langle x, x\rangle \leq \langle S_\Lambda x, x\rangle \leq B'\langle x, x\rangle.$$

Finally, for all $x \in U$, we have

$$A'\langle x, x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) \leq B'\langle x, x\rangle.$$

This completes the proof. □

Theorem 0.18. *If* $T : \oplus_{w\in\Omega} V_w \to U$*, defined by* $T(\{x_w\}_{w\in\Omega}) = \int_\Omega v_w P_{H_w} \Lambda_w^* x_w d\mu(w)$*, is well-defined and surjective, then* $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ *is a continuous g-fusion frame for* U*.*

Proof. We have, for all $x \in U$,

$$\begin{aligned}\|\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w)\| &= \|\int_\Omega \langle x, v_w^2 P_{H_w} \Lambda_w^* \Lambda_w P_{H_w} x\rangle d\mu(w)\| \\ &= \|\langle x, \int_\Omega v_w^2 P_{H_w} \Lambda_w^* \Lambda_w P_{H_w} x d\mu(w)\rangle\| \\ &\le \|x\| \|\int_\Omega v_w^2 P_{H_w} \Lambda_w^* \Lambda_w P_{H_w} x d\mu(w)\| \\ &\le \|x\| \|T(\{v_w \Lambda_w P_{H_w} x\}_{w\in\Omega})\| \\ &\le \|x\| \|T\| \|\{v_w \Lambda_w P_{H_w} x\}_{w\in\Omega}\| \\ &= \|x\| \|T\| \|\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w)\|^{\frac{1}{2}}.\end{aligned}$$

So

$$||\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w)|| \le ||T||^2 ||x||^2. \tag{0.6}$$

On the other hand, since T is surjective, by Lemma 0.6, T^* is bounded below and hence T^* is injective and so $T^* : U \to \mathcal{R}(T^*)$ is invertible. So for al $x \in U$, $(T^*_{/\mathcal{R}(T^*)})^{-1} T^* x = x$, which implies $||x||^2 \le ||(T^*_{/\mathcal{R}(T^*)})^{-1}||^2 ||T^* x||^2$. Thus

$$||(T^*_{/\mathcal{R}(T^*)})^{-1}||^{-2} ||x||^2 \le ||\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w)||. \tag{0.7}$$

From (0.6) and (0.7), $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a continuous g-fusion frame for U. □

Theorem 0.19. *Let* $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ *and* $\Gamma = \{H_w, \Gamma_w, v_w\}_{w\in\Omega}$ *be two Bessel sequences for* U *with frame bounds* B_1 *and* B_2*, respectively. The operator* Q *on* U*, defined by* $Q(x) = \int_\Omega v_w^2 P_{H_w} \Gamma_w^* \Lambda_w P_{H_w} x d\mu(w)$*, is bounded.*

Proof. Let $x \in U$. Then

$$\begin{aligned}\|Q(x)\| = \sup_{\|y\|=1} \|\langle Q(x), y\rangle\| &= \sup_{\|y\|=1} \|\langle \int_\Omega v_w^2 P_{H_w}\Gamma_w^*\Lambda_w P_{H_w} x d\mu(w), y\rangle\| \\ &= \sup_{\|y\|=1} \|\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Gamma_w P_{H_w} y\rangle d\mu(w)\| \\ \leq \sup_{\|y\|=1} \big(\|\int_\Omega v_w^2\langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w)\|^{\frac{1}{2}} &\|\int_\Omega v_w^2\langle \Gamma_w P_{H_w} y, \Gamma_w P_{H_w} y\rangle d\mu(w)\|^{\frac{1}{2}}\big) \\ &\leq (B_1B_2)^{\frac{1}{2}}\|x\|.\end{aligned}$$

This completes the proof. □

Theorem 0.20. *Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous g-fusion frame for U and $\Gamma = \{H_w, \Gamma_w, v_w\}_{w\in\Omega}$ be a continuous g-fusion Bessel sequence for U. If the operator Q, defined in Theorem 0.19, is surjective, then Γ is a continuous g-fusion frame for U.*

Proof. Suppose that Λ is a continuous g-fusion frame for U with synthesis operator T_Λ. Since Γ is a continuous g-fusion Bessel sequence for U, we define the synthesis operator $T_\Gamma : \oplus_{w\in\Omega} V_w \to U$ by $T_\Gamma(\{x_w\}_{w\in\Omega}) = \int_\Omega v_w P_{H_w}\Gamma_w^* x_w d\mu(w)$.

We have, for all $x \in U$,

$$\begin{aligned}Q(x) &= \int_\Omega v_w^2 P_{H_w}\Gamma_w^*\Lambda_w P_{H_w} x_w d\mu(w) = \int_\Omega v_w P_{H_w}\Gamma_w^*(v_w\Lambda_w P_{H_w} x) d\mu(w) \\ &= T_\Gamma T_\Lambda^* x.\end{aligned}$$

Then $Q = T_\Gamma T_\Lambda^*$. Since Q is surjective, there exists $x \in U$ such that $y = Q(x) = T_\Gamma(T_\Lambda^* x)$ and $T_\Lambda^* x \in \oplus_{w\in\Omega} V_w$ and hence T_Γ is surjective. So by Theorem 0.18, Γ is a continuous g-fusion frame for U. □

Theorem 0.21. *Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous g-fusion frame for U with frame bounds A and B. If $\theta \in End_{\mathcal{A}}^*(U)$ is injective and has a closed range and $\theta P_{H_w} = P_{H_w}\theta$ for all $w \in \Omega$, then $\{H_w, \Lambda_w\theta, v_w\}_{w\in\Omega}$ is a continuous g-fusion frame for U.*

Proof. Since Λ is a continuous g-fusion frame, for all $x \in U$,

$$A\langle x, x\rangle \leq \int_\Omega v_w^2\langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) \leq B\langle x, x\rangle.$$

We have, for all $x \in U$,

$$\begin{aligned}\int_\Omega v_w^2 \langle \Lambda_w \theta P_{H_w} x, \Lambda_w \theta P_{H_w} x\rangle d\mu(w) &= \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} \theta x, \Lambda_w P_{H_w} \theta x\rangle d\mu(w) \\ &\leq B\langle \theta x, \theta x\rangle \\ &\leq B\|\theta\|^2 \langle x, x\rangle. \qquad (0.8)\end{aligned}$$

On the other hand, for all $x \in U$,

$$\begin{aligned}\int_\Omega v_w^2 \langle \Lambda_w \theta P_{H_w} x, \Lambda_w \theta P_{H_w} x\rangle d\mu(w) &= \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} \theta x, \Lambda_w P_{H_w} \theta x\rangle d\mu(w) \\ &\geq A\langle \theta x, \theta x\rangle. \qquad (0.9)\end{aligned}$$

Since θ is injective and has a closed range,

$$\|(\theta^*\theta)^{-1}\|^{-1}\langle x, x\rangle \leq \langle \theta^*\theta x, x\rangle, \qquad \forall x \in U. \qquad (0.10)$$

By (0.9) and (0.10), we have

$$A\|(\theta^*\theta)^{-1}\|^{-1}\langle x, x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w \theta P_{H_w} x, \Lambda_w \theta P_{H_w} x\rangle d\mu(w), \qquad \forall x \in U. \qquad (0.11)$$

From (0.8) and (0.11), we conclude that $\{H_w, \Lambda_w\theta, v_w\}_{w\in\Omega}$ is a continuous g-fusion frame for U. □

Theorem 0.22. *Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous g-fusion frame for U with frame bounds A and B. If $\theta \in End_{\mathcal{A}}^*(U, V_w)$ is injective and has a closed range, then $\{H_w, \theta\Lambda_w, v_w\}_{w\in\Omega}$ is a continuous g-fusion frame for U.*

Proof. We have, for all $x \in U$,

$$A\langle x, x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) \leq B\langle x, x\rangle.$$

Since θ is injective and has a closed range, for all $x \in U$,

$$\begin{aligned}\|(\theta^*\theta)^{-1}\|^{-1}\langle v_w\Lambda_w P_{H_w} x, v_w\Lambda_w P_{H_w} x\rangle &\leq \langle \theta^*\theta v_w\Lambda_w P_{H_w} x, v_w\Lambda_w P_{H_w} x\rangle \\ &\leq \|\theta\|^2\langle v_w\Lambda_w P_{H_w} x, v_w\Lambda_w P_{H_w} x\rangle.\end{aligned}$$

So, for all $x \in U$,

$$\|(\theta^*\theta)^{-1}\|^{-1}\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) \leq \int_\Omega v_w^2 \langle \theta^*\theta\Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w)$$

and

$$\int_\Omega v_w^2 \langle \theta^*\theta\Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) \leq \|\theta\|^2 \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w).$$

Hence for all $x \in U$,

$$A||(\theta^*\theta)^{-1}||^{-1}\langle x, x\rangle \leq \int_\Omega v_w^2\langle \theta\Lambda_w P_{H_w}x, \theta\Lambda_w P_{H_w}x\rangle d\mu(w) \leq B||\theta||^2\langle x, x\rangle.$$

Therefore, $\{H_w, \theta\Lambda_w, v_w\}_{w\in\Omega}$ is a continuous g-fusion frame for U. □

We give a relationship between a continuous frame and continuous g-fusion frame in Hilbert C^*-modules.

Theorem 0.23. *Let $w \in \Omega$, $\Lambda_w \in End_{\mathcal{A}}^*(U, \oplus_{w\in\Omega}V_w)$ and $\{y_{w,v}\}_{v\in\Omega_w}$ be a continuous frame for V_w with frame bounds C_w, D_w such that there exist C, $D > 0$ for which $C \leq C_w$ and $D_w \leq D$, then the following conditions are equivalent:*

(1) *$\{v_w P_{H_w}\Lambda_w^* y_{w,v}\}_{v\in\Omega_w}$ is a continuous frame for U.*
(2) *$\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a continuous g-fusion frame for U.*

Proof. Since $\{y_{w,v}\}_{v\in\Omega_w}$ is a continuous frame for V_w, for all $x \in U$,

$$\begin{aligned} C_w\langle v_w\Lambda_w P_{H_w}x, v_w\Lambda_w P_{H_w}x\rangle &\leq \int_{\Omega_w}\langle v_w\Lambda_w P_{H_w}x, y_{w,v}\rangle\langle y_{w,v}, v_w\Lambda_w P_{H_w}x\rangle d\mu(v) \\ &\leq D_w\langle v_w\Lambda_w P_{H_w}x, v_w\Lambda_w P_{H_w}x\rangle. \end{aligned}$$

Thus

$$\begin{aligned} C_w\langle v_w\Lambda_w P_{H_w}x, v_w\Lambda_w P_{H_w}x\rangle &\leq \int_{\Omega_w}\langle x, v_w P_{H_w}\Lambda_w^* y_{w,v}\rangle\langle v_w P_{H_w}\Lambda_w^* y_{w,v}, x\rangle d\mu(v) \\ &\leq D_w\langle v_w\Lambda_w P_{H_w}x, v_w\Lambda_w P_{H_w}x\rangle. \end{aligned}$$

Hence

$$\begin{aligned} C\langle v_w\Lambda_w P_{H_w}x, v_w\Lambda_w P_{H_w}x\rangle &\leq \int_{\Omega_w}\langle x, v_w P_{H_w}\Lambda_w^* y_{w,v}\rangle\langle v_w P_{H_w}\Lambda_w^* y_{w,v}, x\rangle d\mu(v) \\ &\leq D\langle v_w\Lambda_w P_{H_w}x, v_w\Lambda_w P_{H_w}x\rangle. \end{aligned}$$

So

$$\begin{aligned} C\int_\Omega v_w^2\langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w) &\leq \int_\Omega\int_{\Omega_w}\langle x, v_w P_{H_w}\Lambda_w^* y_{w,v}\rangle\langle v_w P_{H_w}\Lambda_w^* y_{w,v}, x\rangle d\mu(v)d\mu(w) \\ &\leq D\int_\Omega v_w^2\langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w). \qquad (0.12) \end{aligned}$$

Suppose that, $\{v_w P_{H_w}\Lambda_w^* y_{w,v}\}_{v\in\Omega_w}$ is a continuous frame for U with frame bounds C' and D'. Then for all $x \in U$,

$$C'\langle x, x\rangle \leq \int_\Omega\int_{\Omega_w}\langle x, v_w P_{H_w}\Lambda_w^* y_{w,v}\rangle\langle v_w P_{H_w}\Lambda_w^* y_{w,v}, x\rangle d\mu(v)d\mu(w) \leq D'\langle x, x\rangle. \qquad (0.13)$$

By (0.12) and (0.13), we have

$$C\int_{\Omega} v_w^2\langle\Lambda_w P_{H_w}x,\Lambda_w P_{H_w}x\rangle d\mu(w)\leq D^{'}\langle x,x\rangle$$

and

$$C^{'}\langle x,x\rangle\leq D\int_{\Omega} v_w^2\langle\Lambda_w P_{H_w}x,\Lambda_w P_{H_w}x\rangle d\mu(w).$$

Therefore,

$$D^{-1}C^{'}\langle x,x\rangle\leq\int_{\Omega} v_w^2\langle\Lambda_w P_{H_w}x,\Lambda_w P_{H_w}x\rangle d\mu(w)\leq C^{-1}D^{'}\langle x,x\rangle,\qquad\forall x\in U.$$

Thus we conclude that $\{H_w,\Lambda_w,v_w\}_{w\in\Omega}$ is a continuous g-fusion frame for U.

Conversely, suppose that $\{H_w,\Lambda_w,v_w\}_{w\in\Omega}$ is a continuous g-fusion frame for U with frame bounds $C^{'}$ and $D^{'}$. Then for all $x\in U$, we have

$$C^{'}\langle x,x\rangle\leq\int_{\Omega} v_w^2\langle\Lambda_w P_{H_w}x,\Lambda_w P_{H_w}x\rangle d\mu(w)\leq D^{'}\langle x,x\rangle.$$

So by (0.12), for all $x\in U$, we have

$$CC^{'}\langle x,x\rangle\leq\int_{\Omega}\int_{\Omega_w}\langle x,v_w P_{H_w}\Lambda_w^* y_{w,v}\rangle\langle v_w P_{H_w}\Lambda_w^* y_{w,v},x\rangle d\mu(v)d\mu(w)$$

and

$$\int_{\Omega}\int_{\Omega_w}\langle x,v_w P_{H_w}\Lambda_w^* y_{w,v}\rangle\langle v_w P_{H_w}\Lambda_w^* y_{w,v},x\rangle d\mu(v)d\mu(w)\leq DD^{'}\langle x,x\rangle.$$

We conclude that $\{v_w P_{H_w}\Lambda_w^* y_{w,v}\}_{v\in\Omega_w}$ is a continuous frame for U. □

Corollary 0.24. *Let $w\in\Omega$, $\Lambda_w\in End_{\mathcal{A}}^*(U,V_w)$ and $\{y_{w,v}\}_{v\in\Omega_w}$ be a Parseval continuous frame for V_w. Then the continuous g-fusion frame operator of $\Lambda=\{H_w,\Lambda_w,v_w\}_{w\in\Omega}$ is a continuous frame operator of $\{v_w P_{H_w}\Lambda_w^* y_{w,v}\}_{v\in\Omega_w}$.*

Proof. Let $x\in U$ and $y\in V_w$. Then

$$\begin{aligned}\langle v_w P_{H_w}\Lambda_w^* y,x\rangle&=\langle y,v_w\Lambda_w P_{H_w}x\rangle\\&=\langle\int_{\Omega_w}\langle y,y_{w,v}\rangle y_{w,v}d\mu(v),v_w\Lambda_w P_{H_w}x\rangle\\&=\int_{\Omega_w}\langle y,y_{w,v}\rangle\langle y_{w,v},v_w\Lambda_w P_{H_w}x\rangle d\mu(v)\\&=\langle\int_{\Omega_w}\langle y,y_{w,v}\rangle v_w P_{H_w}\Lambda_w^* y_{w,v}d\mu(v),x\rangle.\end{aligned}$$

So

$$v_w P_{H_w}\Lambda_w^* y=\int_{\Omega_w}\langle y,y_{w,v}\rangle v_w P_{H_w}\Lambda_w^* y_{w,v}d\mu(v).$$

Hence

$$\int_{\Omega} v_w^2 P_{H_w}\Lambda_w^*\Lambda_w P_{H_w} x d\mu(w) = \int_{\Omega}\int_{\Omega_w} v_w^2 \langle \Lambda_w P_{H_w} x, y_{w,v}\rangle P_{H_w}\Lambda_w^* y_{w,v} d\mu(v) d\mu(w).$$

Therefore, the operator frame of $\{v_w P_{H_w}\Lambda_w^* y_{w,v}\}_{v\in\Omega_w}$ is the continuous g-fusion frame operator of Λ. □

Theorem 0.25. *Let $\{U, \mathcal{A}, \langle .,.\rangle_{\mathcal{A}}\}$ and $\{U, \mathcal{B}, \langle .,.\rangle_{\mathcal{B}}\}$ be two Hilbert C^*-modules and $\phi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism and θ be a map on U such that $\phi(\langle x, y\rangle_{\mathcal{A}}) = \langle \theta x, \theta y\rangle_{\mathcal{B}}$ for all x, $y \in U$. Suppose that θ is surjective and $\theta\Lambda_w P_{H_w} = \Lambda_w P_{H_w}\theta$ for all $w \in \Omega$. If $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a continuous g-fusion frame for $\{U, \mathcal{A}, \langle .,.\rangle_{\mathcal{A}}\}$ with frame bounds A and B, then $\{H_w, \Lambda_w, \phi(v_w)\}_{w\in\Omega}$ is a continuous g-fusion frame for $\{U, \mathcal{B}, \langle .,.\rangle_{\mathcal{B}}\}$ with frame bounds A and B. Moreover $\phi(\langle S_{\mathcal{A}} x, y\rangle_{\mathcal{A}}) = \langle S_{\mathcal{B}}\theta x, \theta y\rangle_{\mathcal{B}}$.*

Proof. Let $y \in U$. Since θ is surjective on U, there exists $x \in U$ such that $y = \theta(x)$, and so we have

$$A\langle x, x\rangle_{\mathcal{A}} \leq \int_{\Omega} v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle_{\mathcal{A}} d\mu(w) \leq B\langle x, x\rangle_{\mathcal{A}}.$$

By definition of $*$-homomorphism, we have

$$\phi(A\langle x, x\rangle_{\mathcal{A}}) \leq \phi\big(\int_{\Omega} v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle_{\mathcal{A}} d\mu(w)\big) \leq \phi(B\langle x, x\rangle_{\mathcal{A}}).$$

Hence

$$A\langle \theta x, \theta x\rangle_{\mathcal{B}} \leq \int_{\Omega} \phi(v_w)^2 \langle \theta\Lambda_w P_{H_w} x, \theta\Lambda_w P_{H_w} x\rangle_{\mathcal{B}} d\mu(w) \leq B\langle \theta x, \theta x\rangle_{\mathcal{B}}.$$

Thus

$$A\langle y, y\rangle_{\mathcal{B}} \leq \int_{\Omega}$$

Moreover, let $x,\, y \in U$. then

$$\begin{aligned}
\phi(\langle S_{\mathcal{A}}x, y\rangle_{\mathcal{A}}) &= \phi\big(\langle \int_{\Omega} v_w^2 P_{H_w}\Lambda_w^*\Lambda_w P_{H_w} x d\mu(w), y\rangle_{\mathcal{A}}\big)\\
&= \phi\big(\int_{\Omega} \langle v_w^2 P_{H_w}\Lambda_w^*\Lambda_w P_{H_w} x, y\rangle_{\mathcal{A}} d\mu(w)\big)\\
&= \int_{\Omega} \phi\big(\langle v_w^2 P_{H_w}\Lambda_w^*\Lambda_w P_{H_w} x, y\rangle_{\mathcal{A}}\big) d\mu(w)\\
&= \int_{\Omega} \phi(v_w^2)\langle \theta\Lambda_w P_{H_w} x, \theta\Lambda_w P_{H_w} y\rangle_{\mathcal{B}} d\mu(w)\\
&= \int_{\Omega} \phi(v_w)^2\langle \Lambda_w P_{H_w} \theta x, \Lambda_w P_{H_w} \theta y\rangle_{\mathcal{B}} d\mu(w)\\
&= \int_{\Omega} \phi(v_w)^2\langle P_{H_w}\Lambda_w^*\Lambda_w P_{H_w} \theta x, \theta y\rangle_{\mathcal{B}} d\mu(w)\\
&= \langle \int_{\Omega} \phi(v_w)^2 P_{H_w}\Lambda_w^*\Lambda_w P_{H_w} \theta x d\mu(w), \theta y\rangle_{\mathcal{B}}\\
&= \langle S_{\mathcal{B}}\theta x, \theta y\rangle_{\mathcal{B}}.
\end{aligned}$$

This completes the proof. □

3. Continuous K-g-fusion frame in Hilbert C^*-modules

We begin this section with the following lemma.

Lemma 0.26. *Let $\{H_w\}_{w\in\Omega}$ be a sequence of orthogonally complemented closed submodules of U and $T \in End_{\mathcal{A}}^*(U)$ is invertible. If $T^*TH_w \subseteq H_w$ for all $w \in \Omega$, then $\{TH_w\}_{w\in\Omega}$ is a sequence of orthogonally complemented closed submodules and $P_{H_w}T^* = P_{H_w}T^*P_{TH_w}$.*

Proof. Firstly, for each $w \in \Omega$, $T : H_w \to TH_w$ is invertible and so each TH_w is a closed submodule of U. We will show that $U = TH_w \oplus T(H_w^\perp)$. Since $U = TU$, for each $x \in U$, there exists $y \in U$ sutch that $x = Ty$. On the other hand, $y = u + v$ for some $u \in H_w$ and $v \in H_w^\perp$. Hence $x = Tu + Tv$, where $Tu \in TH_w$ and $Tv \in T(H_w^\perp)$. It is easy to show that $TH_w \cap T(H_w^\perp) = (0)$. So $U = TH_w \oplus T(H_w^\perp)$. Hence for every $y \in H_w$, $h \in H_w^\perp$ we have $T^*Ty \in H_w$ and therefore $\langle Ty, Th\rangle = \langle T^*Ty, h\rangle = 0$ and so $T(H_w^\perp) \subset (TH_w)^\perp$ and consequently $T(H_w^\perp) = (TH_w)^\perp$ which implies that TH_w is orthogonally complemented.

Let $x \in U$. Then we have $x = P_{TH_w}x + y$ for some $y \in (TH_w)^\perp$. Then $T^*x = T^*P_{TH_w}x + T^*y$. Let $v \in H_w$. Then $\langle T^*y, v\rangle = \langle y, Tv\rangle = 0$ and so $T^*y \in H_w^\perp$. Thus we have $P_{H_w}T^*x = P_{H_w}T^*P_{TH_w}x + P_{H_w}T^*y$ and so $P_{H_w}T^*x = P_{H_w}T^*P_{TH_w}x$, which implies that for all $w \in \Omega$ we have $P_{H_w}T^* = P_{H_w}T^*P_{TH_w}$. □

Definition 0.27. Let $K \in End^*_{\mathcal{A}}(U)$. Let $\{H_w\}_{w\in\Omega}$ be a sequence of closed submodules orthogonally complemented in U, P_{H_w} be the orthogonal projection from U to H_w, $\Lambda_w \in End^*_{\mathcal{A}}(U, V_w)$ for all $w \in \Omega$ and $\{v_w\}_{w\in\Omega}$ be a family of weights in $\mathcal{A}$, i.e., each v_w is a positive invertible element from the center of $\mathcal{A}$. Then we say that $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a continuous K-g-fusion frame for U if

(1) for each $x \in U$, $\{P_{H_w}x\}_{w\in\Omega}$ is measurable;
(2) for each $x \in U$, the function $\tilde{\Lambda} : \Omega \to V_w$ defined by $\tilde{\Lambda}(w) = \Lambda_w x$ is measurable;
(3) there exist $0 < A \le B < \infty$ such that

$$A\langle K^*x, K^*x\rangle \le \int_\Omega v_w^2 \langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w) \le B\langle x, x\rangle, \qquad \forall x \in U. \tag{0.14}$$

We call A and B lower and upper frame bounds of a continuous K-g-fusion frame, respectively. If the left-hand inequality of (0.14) is an equality, then we say that $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a tight continuous K-g-fusion frame.

If $A = B = 1$, then we say that $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a Parseval continuous K-g-fusion frame for U.

If the right-hand inequality of (0.14) holds, then $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is called a continuous g-fusion Bessel sequence with a bound B for U.

Proposition 0.28. *Let $K \in End^*_{\mathcal{A}}(U)$.*
Every continuous g-fusion frame for U is a continuous K-g-fusion frame for U.

Proof. Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous g-fusion frame for U. Then for all $x \in U$,

$$A\langle x, x\rangle \le \int_\Omega v_w^2 \langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w) \le B\langle x, x\rangle.$$

And we have, for all $x \in U$,

$$\langle K^*x, K^*x\rangle \le ||K||^2\langle x, x\rangle \implies ||K||^{-2}\langle K^*x, K^*x\rangle \le \langle x, x\rangle.$$

So

$$A||K||^{-2}\langle K^*x, K^*x\rangle \le \int_\Omega v_w^2 \langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w) \le B\langle x, x\rangle.$$

Therefore, Λ is a continounus K-g-fusion frame for U. □

Remark 0.29. Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous g-fusion Bessel sequence for U with continuous g-fusion frame operator S_Λ. If Λ is a continuous K-g-fusion frame for U with frame bounds A and B, then we have

$$AKK^* \leq S_\Lambda \leq BI_U. \tag{0.15}$$

Theorem 0.30. *Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous g-fusion Bessel sequence for U with continuous g-fusion frame operator S_Λ for U. Then Λ is a continuous K-g-fusion frame for U if and only if there exists a constant $A > 0$ such that $AKK^* \leq S_\Lambda$.*

Proof. Let Λ be a continuous K-g-fusion frame for U. Then from (0.15) we have the result.

Conversely, assume that there exists a constant $A > 0$ such that $AKK^* \leq S_\Lambda$. Since Λ is a continuous g-fusion Bessel sequence for U, $AKK^* \leq S_\Lambda \leq BI_U$. So Λ is a continuous K-g-fusion frame for U. □

Theorem 0.31. *Let $K \in End^*_{\mathcal{A}}(U)$ and $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous g-fusion Bessel sequence for U with frame operator S_Λ. Suppose that $\overline{\mathcal{R}(S_\Lambda^{\frac{1}{2}})}$ is orthogonally complemented. Then Λ is a continuous K-g-fusion frame for U if and only if $K = S_\Lambda^{\frac{1}{2}}Q$ for some $Q \in End^*_{\mathcal{A}}(U)$.*

Proof. Since the operator frame S_Λ is positive self-adjoint, so is also $S_\Lambda^{\frac{1}{2}}$.

Assume that Λ is a continuous K-g-fusion frame for U. Then there exists $A > 0$ such that $KK^* \leq \frac{1}{A}S_\Lambda^{\frac{1}{2}}S_\Lambda^{\frac{1}{2}}$, and by Lemma 0.9, there exists $Q \in End^*_{\mathcal{A}}(U)$ such that $K = S_\Lambda^{\frac{1}{2}}Q$.

Conversely, suppose that there exists $Q \in End^*_{\mathcal{A}}(U)$ such that $K = S_\Lambda^{\frac{1}{2}}Q$. Then by Lemma 0.9, there exists $\lambda > 0$ such that $KK^* \leq \lambda S_\Lambda^{\frac{1}{2}}S_\Lambda^{\frac{1}{2}} = \lambda S_\Lambda$ and so $\frac{1}{\lambda}KK^* \leq S_\Lambda$. Hence Λ is a continuous K-g-fusion frame for U. □

Theorem 0.32. *Let $T \in End^*_{\mathcal{A}}(U)$ be an invertible operator on U and $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous K-g-fusion frame for U with frame bounds A and B for some $K \in End^*_{\mathcal{A}}(U)$. Then $\Gamma = \{TH_w, \Lambda_w P_{H_w} T^*, v_w\}_{w\in\Omega}$ is a continuous TKT^*-g-fusion frame for U.*

Proof. Let $x \in U$. By Lemma 0.26,

$$\begin{aligned}\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* P_{TH_w} x, \Lambda_w P_{H_w} T^* P_{TH_w} x\rangle d\mu(w) &= \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* x, \Lambda_w P_{H_w} T^* x\rangle d\mu(w) \\ &\leq B\langle T^* x, T^* x\rangle \\ &\leq B\|T\|^2 \langle x, x\rangle. \end{aligned} \tag{0.16}$$

On the other hand, for all $x \in U$,

$$\begin{aligned} A\langle (TKT^*)^* x, (TKT^*)^* x\rangle &= A\langle TK^*T^* x, TK^*T^* x\rangle \\ &\leq A\|T\|^2 \langle K^*T^* x, K^*T^* x\rangle \\ &\leq \|T\|^2 \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* x, \Lambda_w P_{H_w} T^* x\rangle d\mu(w) \\ &= \|T\|^2 \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* x, \Lambda_w P_{H_w} T^* x\rangle d\mu(w). \end{aligned}$$

So

$$A\|T\|^{-2} \langle (TKT^*)^* x, (TKT^*)^* x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* P_{TH_w} x, \Lambda_w P_{H_w} T^* P_{TH_w} x\rangle d\mu(w). \tag{0.17}$$

From (0.16) and (0.17), we have Γ is a continuous TKT^*-g-fusion frame for U. □

Theorem 0.33. *If $\{TH_w, \Lambda_w P_{H_w} T^*, v_w\}_{w\in\Omega}$ is a continuous K-g-fusion frame for U with frame bounds A and B, then $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a continuous $T^{-1}KT$-g-fusion frame for U.*

Proof. Let $x \in U$. By Lemma 0.26, we have

$$\begin{aligned}\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) &= \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^*(T^*)^{-1} x, \Lambda_w P_{H_w} T^*(T^*)^{-1} x\rangle d\mu(w) \\ &= \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* P_{H_w} (T^*)^{-1} x, \Lambda_w P_{H_w} T^* P_{H_w} (T^*)^{-1} x\rangle d\mu(w) \\ &\leq B\langle (T^*)^{-1} x, (T^*)^{-1} x\rangle \\ &\leq B\|(T^*)^{-1}\|\langle x, x\rangle. \end{aligned} \tag{0.18}$$

Also we have, for all $x \in U$,

$$\begin{aligned}A\langle (T^{-1}KT)^*x, (T^{-1}KT)^*x\rangle &= A\langle T^*K^*(T^{-1})^*x, T^*K^*(T^{-1})^*x\rangle \\ &\leq A||T||^2\langle K^*(T^{-1})^*x, K^*(T^{-1})^*x\rangle \\ &\leq ||T||^2 \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* P_{TH_w}(T^{-1})^*x, \Lambda_w P_{H_w} T^* P_{TH_w}(T^{-1})^*x\rangle d\mu(w) \\ &\leq ||T||^2 \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w).\end{aligned}$$

Hence

$$A||T||^{-2}\langle (T^{-1}KT)^*x, (T^{-1}KT)^*x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^*x, \Lambda_w P_{H_w} T^*x\rangle d\mu(w). \tag{0.19}$$

From (0.18) and (0.19), we conclude that $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a continuous $T^{-1}KT$-g-fusion frame for U. □

Theorem 0.34. *Let $K \in End_{\mathcal{A}}^*(U)$ be an invertible operator, $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous g-fusion frame for U with frame bounds A, B and S_Λ be an associated continuous g-fusion frame operator. Then $\{KS_\Lambda^{-1}H_w, \Lambda_w P_{H_w} S_\Lambda^{-1}K^*, v_w\}_{w\in\Omega}$ is a continuous K-g-fusion frame for U.*

Proof. Put $T = KS_\Lambda^{-1}$, which is invertible and $T^* = (KS_\Lambda^{-1})^* = S_\Lambda^{-1}K^*$. Then by Lemma 0.26, for all $x \in U$,

$$\begin{aligned}\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* P_{TH_w} x, \Lambda_w P_{H_w} T^* P_{TH_w} x\rangle d\mu(w) &= \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^*x, \Lambda_w P_{H_w} T^*x\rangle d\mu(w) \\ &\leq B\langle T^*x, T^*x\rangle \\ &\leq B||T||^2\langle x, x\rangle \qquad (0.20)\end{aligned}$$

We have, for all $x \in U$,

$$\begin{aligned}\langle K^*x, K^*x\rangle = \langle S_\Lambda S_\Lambda^{-1}K^*x, S_\Lambda S_\Lambda^{-1}K^*x\rangle &\leq ||S_\Lambda||^2\langle S_\Lambda^{-1}K^*x, S_\Lambda^{-1}K^*x\rangle \\ &\leq B^2\langle S_\Lambda^{-1}K^*x, S_\Lambda^{-1}K^*x\rangle\end{aligned}$$

and

$$\begin{aligned}\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* P_{TH_w} x, \Lambda_w P_{H_w} T^* P_{TH_w} x\rangle d\mu(w) &\geq A\langle S_\Lambda^{-1}K^*x, S_\Lambda^{-1}K^*x\rangle \\ &\geq \frac{A}{B^2}\langle K^*x, K^*x\rangle.\end{aligned}$$

Therefore, $\{KS_\Lambda^{-1}H_w, \Lambda_w P_{H_w} S_\Lambda^{-1}K^*, v_w\}_{w\in\Omega}$ is a K-g-fusion frame for U. □

Theorem 0.35. *Let $K \in End_{\mathcal{A}}^{*}(U)$ and $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous K-g-fusion frame for U with frame bounds A and B. Suppose that $G \in End_{\mathcal{A}}^{*}(U)$, $\mathcal{R}(G) \subset \mathcal{R}(K)$ and $\overline{\mathcal{R}(K^*)}$ is orthogonally complemented. Then Λ is a continuous G-g-fusion frame for U.*

Proof. Since $\mathcal{R}(G) \subseteq \mathcal{R}(K)$ and $\overline{\mathcal{R}(K^*)}$ is orthogonally complemented, by Lemma 0.9, there exists $\lambda > 0$ such that $GG^* \leq \lambda KK^*$ amd hence

$$\frac{A}{\lambda}\langle G^*x, G^*x\rangle \leq A\langle K^*x, K^*x\rangle \leq \langle S_\Lambda x, x\rangle \leq B\langle x, x\rangle, \qquad \forall x \in U.$$

So $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a continuous G-g-fusion frame for U. □

Theorem 0.36. *Let $K \in End_{\mathcal{A}}^{*}(U)$ and $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous g-fusion Bessel sequence for U with synthesis operator T_Λ. Suppose that $\overline{\mathcal{R}(T_\Lambda^*)}$ and $\overline{\mathcal{R}(K^*)}$ are orthogonally complemented. Then the following statements hold:*

1. *If Λ is a tight K-g-fusion frame for U, then $\mathcal{R}(K) = \mathcal{R}(T_\Lambda)$.*
2. *$\mathcal{R}(K) = \mathcal{R}(T_\Lambda)$ if and only if there exist two constants A and B such that*

$$A\langle K^*x, K^*x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) \leq B\langle K^*x, K^*x\rangle, \qquad \forall x \in U.$$

Proof. (1) Suppose that Λ is a continuous tight K-g-fusion frame for U. Then for all $x \in U$,

$$A\langle K^*x, K^*x\rangle = \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) = \langle T_\Lambda^* x, T_\Lambda^* x\rangle.$$

So

$$A\langle KK^*x, x\rangle = \langle T_\Lambda T_\Lambda^* x, x\rangle$$

and hence

$$AKK^* = T_\Lambda T_\Lambda^*.$$

By Lemma 0.9, we have $\mathcal{R}(T_\Lambda) = \mathcal{R}(K)$.

(2) Assume that $\mathcal{R}(K) = \mathcal{R}(T_\Lambda)$. Then by Lemma 0.9, there exist two constants A and B such that

$$AKK^* \leq T_\Lambda T_\Lambda^* \leq BKK^*.$$

Hence

$$A\langle K^*x, K^*x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) \leq B\langle K^*x, K^*x\rangle.$$

Conversely, suppose that there exist two constants A and B such that

$$A\langle K^*x, K^*x\rangle \leq \int_\Omega v_w^2\langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w) \leq B\langle K^*x, K^*x\rangle, \qquad \forall x \in U.$$

Thus

$$A\langle KK^*x, x\rangle \leq \langle T_\Lambda T_\Lambda^* x, x\rangle \leq B\langle KK^*x, x\rangle, \qquad \forall x \in U.$$

So

$$AKK^* \leq T_\Lambda T_\Lambda^* \leq BKK^*.$$

By Lemma 0.9, we have $\mathcal{R}(K) = \mathcal{R}(T_\Lambda)$. □

Theorem 0.37. *Let $K \in End_{\mathcal{A}}^*(U)$ and $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$. Suppose that $T : U \to \oplus_{w\in\Omega}V_w$ is given by $T(x) = \{v_w\Lambda_w P_{H_w}x\}_{w\in\Omega}$. Then Λ is a continuous K-g-fusion frame for U if and only if there exists two constants A and B such that*

$$A||K^*x||^2 \leq ||\int_\Omega v_w^2\langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w)|| \leq B||x||^2, \qquad \forall x \in U. \tag{0.21}$$

Proof. Suppose that Λ is a continuous K-g-fusion frame for U. Then we have (0.21).

Conversely, assume that (0.21) holds. Then we have, for all $x \in U$,

$$\begin{aligned} ||\int_\Omega v_w P_{H_w}\Lambda_w^* x_w d\mu(w)|| &= \sup_{||y||=1} ||\langle \int_\Omega v_w P_{H_w}\Lambda_w^* x d\mu(w), y\rangle|| \\ &= \sup_{||y||=1} ||\int_\Omega \langle x_w, v_w\Lambda_w P_{H_w}y\rangle d\mu(w)|| \\ &\leq \sup_{||y||=1} ||\int_\Omega \langle x_w, x_w\rangle d\mu(w)||^{\frac{1}{2}} ||\int_\Omega v_w^2\langle \Lambda_w P_{H_w}y, \Lambda_w P_{H_w}y\rangle||^{\frac{1}{2}} \\ &\leq \sqrt{B}||\{x_w\}_{w\in\Omega}||. \end{aligned}$$

Thus the series $\int_\Omega v_w P_{H_w}\Lambda_w^* x_w d\mu(w)$ converge in U and we have, for all $x \in U$ and $\{x_w\}_{w\in\Omega} \in \oplus_{w\in\Omega}V_w$,

$$\begin{aligned} \langle Tx, \{x_w\}_{w\in\Omega}\rangle &= \langle \{v_w\Lambda_w P_{H_w}x_w\}_{w\in\Omega}, \{x_w\}_{w\in\Omega}\rangle \\ &= \int_\Omega \langle v_w\Lambda_w P_{H_w}x, x_w\rangle d\mu(w) \\ &= \int_\Omega \langle x, v_w P_{H_w}\Lambda_w^* x_w\rangle d\mu(w) \\ &= \langle x, \int_\Omega v_w P_{H_w}\Lambda_w^* x_w d\mu(w)\rangle. \end{aligned}$$

Hence T is adjointable and so, for all $x \in U$,

$$\langle Tx, Tx\rangle = \int_\Omega v_w^2\langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w) \leq ||T||^2\langle x, x\rangle.$$

On the other hand, from (0.21), we have, for all $x \in U$,

$$||K^*x||^2 \leq \frac{1}{A}||Tx||^2$$

and by Lemma 0.9, there exists a constant $\lambda > 0$ such that $KK^*x \leq \lambda T^*Tx$. Therefore,

$$\frac{1}{\lambda}\langle K^*x, K^*x\rangle \leq \langle Tx, Tx\rangle = \int_\Omega v_w^2\langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w)$$

for all $x \in U$. □

Theorem 0.38. *Let $K_i \in End_{\mathcal{A}}^*(U)$ for all $i \in \{1,..,n\}$ and $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous K_i-g-fusion frame for U with frame bounds A_i and B. Suppose that $T : U \to \oplus_{w\in\Omega}V_w$ is given by $Tx = \{v_w\Lambda_w P_{H_w}x\}_{w\in\Omega}$ and $\overline{\mathcal{R}(T)}$ is orthogonally complemented. Then Λ is a continuous $\sum_{i=1}^n K_i$-g-fusion frame for U.*

Proof. Let $x \in U$. Then

$$\begin{aligned}
||\langle(\sum_{i=1}^n K_i)^*x, (\sum_{i=1}^n K_i)^*x\rangle||^{\frac{1}{2}} &= ||(\sum_{i=1}^n K_i)^*x|| \\
&= ||\sum_{i=1}^n K_i^*x|| \\
&\leq \sum_{i=1}^n ||K_i^*x|| \\
&\leq \sum_{i=1}^n \frac{1}{\sqrt{A_i}}||\int_\Omega v_w^2\langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w)||^{\frac{1}{2}}
\end{aligned}$$

and so

$$(\sum_{i=1}^n \frac{1}{\sqrt{A_i}})^2||\langle(\sum_{i=1}^n K_i)^*x, (\sum_{i=1}^n K_i)^*x\rangle|| \leq ||\int_\Omega v_w^2\langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w)||. \tag{0.22}$$

On the other hand, for all $x \in U$,

$$||\int_\Omega v_w^2\langle \Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w)|| \leq B||x||^2. \tag{0.23}$$

From (0.22) and (0.23), we conclude that Λ is a continuous $\sum_{i=1}^n K_i$-g-fusion frame for U. □

Theorem 0.39. *Let $K_i \in End_{\mathcal{A}}^*(U)$ for all $i \in \{1,..,n\}$ and $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous K_i-g-fusion frame for U with frame bounds A_i and B. Then Λ is a continuous $\prod_{i=1}^n K_i$-g-fusion frame for U.*

Proof. Let $x \in U$. Then

$$\begin{aligned} A_1\langle (\prod_{i=1}^{n} K_i)^* x, (\prod_{i=1}^{n} K_i)^* x\rangle &= A_1 \langle \prod_{i=n}^{1} K_i^* x, \prod_{i=n}^{1} K_i^* x\rangle \\ &\leq A_1 \| \prod_{i=n}^{2} K_i^* \|^2 \langle K_1^* x, K_1^* x\rangle \\ &\leq \prod_{i=n}^{2} \|K_i\|^2 \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) \end{aligned}$$

and hence

$$A_1 (\prod_{i=n}^{2} \|K_i\|^2)^{-1} \langle (\prod_{i=1}^{n} K_i)^* x, (\prod_{i=1}^{n} K_i)^* x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w).$$

And we have, for all $x \in U$,

$$\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) \leq B\langle x, x\rangle.$$

Therefore, Λ is a continuous $\prod_{i=1}^{n} K_i$-g-fusion frame for U. □

4. Perturbation of continuous K-g-fusion frames

Theorem 0.40. *Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous K-g-fusion frame for U with bounds A and B and $\{\Gamma_w\}_{w\in\Omega} \in End_{\mathcal{A}}^*(U, V_w)$. Suppose that*

(1) *for all $x \in U$,*

$$\begin{aligned} &\|\{(v_w\Lambda_w P_{H_w} - z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}\| \\ &\leq \lambda_1 \|\{(v_w\Lambda_w P_{H_w})x\}_{w\in\Omega}\| + \lambda_2 \|\{(z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}\| + \epsilon\|K^* x\|, \end{aligned}$$

where $0 < \lambda_1, \lambda_2 < 1$ and $\epsilon < (1-\lambda_1)\sqrt{A}$;

(2) *$T : U \to \oplus_{w\in\Omega} V_w$ is given by $T(x) = \{z_w\Gamma_w P_{Z_w} x\}_{w\in\Omega}$ and $\overline{\mathcal{R}(T)}$ is orthogonally complemented.*

Then $\{Z_w, \Gamma_w, z_w\}_{w\in\Omega}$ is a continuous K-g-fusion frame for U.

Proof. We have, for all $x \in U$,

$$\begin{aligned} \|\{(z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}\| &\leq \|\{(v_w\Lambda_w P_{H_w} - z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}\| + \|\{v_w\Lambda_w P_{H_w} x\}_{w\in\Omega}\| \\ &\leq \lambda_1 \|\{(v_w\Lambda_w P_{H_w})x\}_{w\in\Omega}\| + \lambda_2 \|\{(z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}\| + \epsilon\|K^* x\| \\ &\qquad + \|\{(v_w\Lambda_w P_{H_w})x\}_{w\in\Omega}\| \\ &\leq (\lambda_1 + 1)\|\{(v_w\Lambda_w P_{H_w})x\}_{w\in\Omega}\| + \lambda_2 \|\{(z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}\| + \epsilon\|K^* x\| \end{aligned}$$

and hence

$$(1-\lambda_2)||\{(z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}|| \leq (1+\lambda_1)||\{(v_w\Lambda_w P_{H_w})x\}_{w\in\Omega}|| + \epsilon||K^*x||.$$

So

$$(1-\lambda_2)||\int_\Omega z_w^2\langle\Gamma_w P_{Z_w}x, \Gamma_w P_{Z_w}x\rangle d\mu(w)||^{\frac{1}{2}} \leq (1+\lambda_1)\sqrt{B}||x|| + \epsilon||K||||x||.$$

Thus

$$||\int_\Omega z_w^2\langle\Gamma_w P_{Z_w}x, \Gamma_w P_{Z_w}x\rangle d\mu(w)|| \leq \left(\frac{(1+\lambda_1)\sqrt{B}+\epsilon||K||}{1-\lambda_2}\right)^2||x||^2. \tag{0.24}$$

On the other hand, for all $x\in U$,

$$\begin{aligned}||\{(z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}|| &\geq ||\{v_w\Lambda_w P_{H_w}x\}_{w\in\Omega}|| - ||\{(v_w\Lambda_w P_{H_w} - z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}|| \\ &\geq ||\{v_w\Lambda_w P_{H_w}x\}_{w\in\Omega}|| - \lambda_1||\{(v_w\Lambda_w P_{H_w})x\}_{w\in\Omega}|| \\ &\qquad - \lambda_2||\{(z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}|| - \epsilon||K^*x|| \\ &\geq (1-\lambda_1)||\{v_w\Lambda_w P_{H_w}x\}_{w\in\Omega}|| - \lambda_2||\{(z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}|| - \epsilon||K^*x||\end{aligned}$$

and so

$$(1+\lambda_2)||\{(z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}|| \geq (1-\lambda_1)||\{v_w\Lambda_w P_{H_w}x\}_{w\in\Omega}|| - \epsilon||K^*x||.$$

Thus

$$||\int_\Omega z_w^2\langle\Gamma_w P_{Z_w}x, \Gamma_w P_{Z_w}x\rangle d\mu(w)||^{\frac{1}{2}} \geq \left(\frac{(1-\lambda_1)\sqrt{A}-\epsilon}{1+\lambda_2}\right)||K^*x||$$

and hence

$$||\int_\Omega z_w^2\langle\Gamma_w P_{Z_w}x, \Gamma_w P_{Z_w}x\rangle d\mu(w)|| \geq \left(\frac{(1-\lambda_1)\sqrt{A}-\epsilon}{1+\lambda_2}\right)^2||K^*x||^2. \tag{0.25}$$

From (0.24) and (0.25), we conclude that $\{Z_w, \Gamma_w, z_w\}_{w\in\Omega}$ is a continuous K-g-fusion frame for U. $\square$

Theorem 0.41. *Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a continuous K-g-fusion frame for U with frame bounds A and B and $\{\Gamma_w\}_{w\in\Omega} \in End_{\mathcal{A}}^*(U, V_w)$. Suppose that*

(1) *there exists $M > 0$ such that, for all $x \in U$,*

$$||\int_\Omega\langle(v_w\Lambda_w P_{H_w} - z_w\Gamma_w P_{Z_w})x, (v_w\Lambda_w P_{H_w} - z_w\Gamma_w P_{Z_w})x\rangle d\mu(w)||$$
$$\leq M\left(||\int_\Omega v_w^2\langle\Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w)||; ||\int_\Omega z_w^2\langle\Gamma_w P_{Z_w}x, \Gamma_w P_{Z_w}x\rangle d\mu(w)||\right);$$

(2) *$T : U \to \oplus_{w\in\Omega}V_w$ is given by $T(x) = \{z_w\Gamma_w P_{Z_w}x\}_{w\in\Omega}$ and $\overline{\mathcal{R}(T)}$ is orthogonally complemented.*

Then $\{Z_w, \Gamma_w, z_w\}_{w\in\Omega}$ *is a continuous* K*-*g*-fusion frame for* U*.*

Proof. For all $x \in U$,

$$\begin{aligned}\sqrt{A}||K^*x|| &\leq ||\int_\Omega v_w^2\langle\Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w)||^{\frac{1}{2}}\\ &= ||\{(v_w\Lambda_w P_{H_w})x\}||\\ &\leq ||\{(v_w\Lambda_w P_{H_w} - z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}|| + ||\{(z_w\Gamma_w P_{Z_w})x\}_{w\in\Omega}||\\ &= ||\int_\Omega\langle(v_w\Lambda_w P_{H_w} - z_w\Gamma_w P_{Z_w})x, (v_w\Lambda_w P_{H_w} - z_w\Gamma_w P_{Z_w})x\rangle d\mu(w)||^{\frac{1}{2}}\\ &\qquad + ||\int_\Omega z_w^2\langle\Gamma_w P_{Z_w}x, \Gamma_w P_{Z_w}x\rangle d\mu(w)||^{\frac{1}{2}}\\ &\leq (1+\sqrt{M})||\int_\Omega z_w^2\langle\Gamma_w P_{Z_w}x, \Gamma_w P_{Z_w}x\rangle d\mu(w)||^{\frac{1}{2}}\end{aligned}$$

and so

$$\frac{\sqrt{A}}{1+\sqrt{M}}||K^*x|| \leq ||\int_\Omega z_w^2\langle\Gamma_w P_{Z_w}x, \Gamma_w P_{Z_w}x\rangle d\mu(w)||^{\frac{1}{2}}. \tag{0.26}$$

On the other hand, for all$x \in U$,

$$\begin{aligned}||\int_\Omega z_w^2\langle\Gamma_w P_{Z_w}x, \Gamma_w P_{Z_w}x\rangle d\mu(w)||^{\frac{1}{2}} &= ||\{z_w\Lambda_w P_{Z_w}x\}_{w\in\Omega}||\\ &\leq ||\{(z_w\Gamma_w P_{Z_w} - v_w\Lambda_w P_{V_w})x\}_{w\in\Omega}|| + ||\{(v_w\Lambda_w P_{H_w})x\}_{w\in\Omega}||\\ &\leq \sqrt{M}||\int_\Omega v_w^2\langle\Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w)||^{\frac{1}{2}} + ||\int_\Omega v_w^2\langle\Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w)||^{\frac{1}{2}}\\ &\leq (\sqrt{M}+1)||\int_\Omega v_w^2\langle\Lambda_w P_{H_w}x, \Lambda_w P_{H_w}x\rangle d\mu(w)||^{\frac{1}{2}}\end{aligned}$$

(0.27)

$$\leq (\sqrt{M}+1)\sqrt{B}||x||.$$

Then from (0.26) and (0.27), we conclude that $\{Z_w, \Gamma_w, z_w\}_{w\in\Omega}$ is a continuous K-g-fusion frame for U. □

5. Conclusion

We introduced the concept of continuous g-fusion frame and K-g-fusion frame in Hilbert C^*-modules. Furthermore, we investigated some properties of them and discussed the perturbation problem for continuous K-g-fusion frames.

Declarations

Availablity of data and materials

Not applicable.

Competing interest

The authors declare that they have no competing interests.

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Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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