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Determinant and inverse of T-Sequence-Sylvester-Kac Matrix

Jeki Saputra[1]*, Amir Kamal Amir[2]*, Andi Muhammad Anwar[3]*

**Departemet of Mathematic, Hasanuddin University, Makassar, Indonesia*

E-mail: *Saputra18h@studentunhas.ac.id[1], amirkamir@science.unhas.ac.id[2], andimuhammadanwar08@gmail.com*

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Abstract

The Sylvester-Kac matrix is also known as the Clement matrix The Sylvester-Kac matrix is widely used and applied both in processing, graphs and other fields. The Sylvester-Kac matrix developed in the paper is the T-Sequence-Sylvester-Kac matrix The calculation of the determinant, and inverse has always been a challenge for mathematicians to find. In this paper will be given the formulation of determinant, and inverse of the T-Sequence-Sylvester-Kac matrix

Keyword : Determinant matrix, Inverse Matrix, Sylvester-Kac matrix

1. INTRODUCTION AND PREMILINARIES

The Sylvester-Kac matrix was first introduced by James J. Sylvester in 1854 [8], then research conducted by Kac found the determinant of the matrix. Some of the developments carried out by W. Chu, et al [2], [3], [4] then R. Bevilaqua, et al [1], C.M. da Fonseca, E. Kilic [5], [6] and by Z. Jiang, et al. [7] regarding the Sylvester-Kac matrix both spectrum, determinant and inverse search.

Definition 1.1. The Sylvester-Kac matrix is a tridiagonal matrix of order n with the main diagonal zero, one of the subdiagonals (1,2,…,n-1), and the other (n-1,n-2,…,1) i.e. [5]

$$\begin{bmatrix} 0 & 1 & & & \\ n-1 & 0 & 2 & & \\ & n-2 & \ddots & \ddots & \\ & & \ddots & 0 & n-1 \\ & & & 1 & 0 \end{bmatrix}. \qquad (1.1)$$

Definition 1.2. The T-Sequence-Sylvester-Kac matrix is a Sylvester-Kac matrix type with the main diagonal a constant and the subdiagonal a sequence which can be written $K_{a,n}^*$ if

$$\left(K_{a,n}^*\right)_{ij} = \begin{cases} t, & i = j, \\ a_i, & j = i+1, 1 \le i \le n-1, \\ a_{n+1-i} & j = i-1,\ \ 2 \le i \le n \\ 0, & \text{else,} \end{cases} \qquad (1.2)$$

where $a_1, a_2, \dots, a_{n-1}$ is a sequence. Especially when $t = 0$ the T-Sequence-Sylvester-Kac matrix is called the Sequence-Sylvester-Kac Matrix.

Note that the T-Sequence-Sylvester-Kac matrix satisfies $K_{a,n}^* = \hat{I}_n K_{a,n}^* \hat{I}_n$ where $\hat{I}_n$ represents the counter identity matrix [7].

2. MAIN RESULTS

In this section we will find the determinant and inverse of the T-Sequence-Sylvester-Kac matrix.

2.1. Determinant of T-Sequence-Sylvester-Kac Matrix

Before giving the determinant of the T-Sequence-Sylvester-Kac matrix, the following theorem will be given:

Theorem 2.1

Suppose defined

$$C_{n,i} = \sum_{|x_k - x_l| > 1, x_j \le n} b_{x_1} b_{x_2} \dots b_{x_i}, \quad x_i = 1,2 \dots, n \qquad (2.1)$$

with $C_{n,0} = 1$ *for* $n = 0,1,2,\dots$ *and* $b_i = a_i a_{n-i}$.

If $p_i = t p_{i-1} - b_{i-1} p_{i-2}$ $(i = 3,4,\dots,)$, *with* $p_1 = t$, *and* $p_2 = t^2 - a_1 a_{n-1} = t^2 - b_1$, *then holds*

$$p_n = \begin{cases} \displaystyle\sum_{i=0}^{\frac{n-1}{2}} (-1)^i C_{n-1,i} t^{n-2i}, & n \text{ odd} \\ \displaystyle\sum_{i=0}^{\frac{n}{2}} (-1)^i C_{n-1,i} t^{n-2i}, & n \text{ even} \end{cases} \qquad (2.2)$$

for $n = 1,2,\dots$

Proof :

By using strong induction, it will be proven that the above statement is true

1. Base step

$$\begin{aligned} P(1) &= p_1 \\ &= \sum_{i=0}^{0} (-1)^i C_{0,i} t^{1-2i} \\ &= (-1)^0 C_{0,0} t^1 \\ &= c \end{aligned}$$

2. Induction step

Assuming that $P(2), P(3), \ldots, P(n)$ is true then it will be proved that $P(n+1)$ is true.

Without of generality, suppose n is even then $n+1$ odd so that from equation 4.1 by substituting $i = n+1$, we get

$$\begin{aligned} p_{n+1} &= tp_n - b_n p_{n-1} \\ &= t \sum_{i=0}^{\frac{n}{2}} (-1)^i C_{n-1,i} c^{n-2i} - b_n \sum_{i=0}^{\frac{n-2}{2}} (-1)^i C_{n-1,i} t^{n-1-2i} \\ &= C_{n-1,0} t^{n+1} + \sum_{i=0}^{\frac{n}{2}} (-1)^i C_{n-1,i} t^{n+1-2i} + \sum_{i=1}^{\frac{n}{2}} (-1)^i b_n C_{n-1,i} t^{n+1-2i} \\ &= C_{n,0} t^{n+1} + \sum_{i=1}^{\frac{n}{2}} (-1)^i C_{n,i} t^{n+1-2i} \\ &= \sum_{i=0}^{\frac{n}{2}} (-1)^i C_{n,i} t^{n+1-2i} \end{aligned}$$

Which of the last statement can be written

$$p_{n^*} = \sum_{i=0}^{\frac{n^*-1}{2}} (-1)^i C_{n^*-1,i} t^{n^*-2i}$$

for $n^* = n+1$ odd (proven).

We will now give the determinant of the T-Sequence-Sylvester-Kac matrix as follows:

Note that the T-Sequence-Sylvester-Kac matrix can be written as

$$K_{a,n}^{*} = \begin{bmatrix} t & a_1 & & & \\ a_{n-1} & t & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_2 & t & a_{n-1} \\ & & & a_1 & t \end{bmatrix}$$

and suppose that $\det(K_{a,n}^{*}) = k_n$ By doing row expansion followed by the last column expansion, we get

$$\begin{aligned} k_n &= tk_{n-1} - a_1 a_{n-1} k_{n-2} \\ &= tk_{n-1} - b_1 k_{n-2} \end{aligned}$$

Note that $k_1 = t, k_2 = t^2 - b_1$, so based on Theorem 2.1 we get that

$$k_n = \begin{cases} \sum_{i=0}^{\frac{n-1}{2}} (-1)^i C_{n-1,i} t^{n-2i}, & n \text{ odd} \\ \sum_{i=0}^{\frac{n}{2}} (-1)^i C_{n-1,i} t^{n-2i}, & n \text{ even} \end{cases} \tag{2. 3}$$

2.2. Inverse of T-Sequence-Sylvester-Kac Matrix

Lemma 2.2

The inverse of the T-Sequence-Sylvester-Kac matrix exists if t is not an eigenvalue of the Sequence-Sylvester-Kac matrix.

So, if t is not an eigenvalue of the Sequence-Sylvester-Kac matrix, then the T-Sequence-Sylvester-Kac matrix has an inverse which will be given by the following Theorem:

Theorem 2.3

Defined $d_0 = 1$, *then the inverse entry of the T-Sequence -Sylvester-Kac matrix is*

$$\left(K_{a,n}^{*}{}^{-1}\right)_{ij} = \begin{cases} \dfrac{d_{i-1} d_{n-i}}{d_n}, & i = j \\ (-1)^{i+j} \dfrac{d_{i-1} d_{n-j}}{d_n} \prod_{k=0}^{j-i-1} a_{i+k}, & i < j \\ (-1)^{i+j} \dfrac{d_{j-1} d_{n-i}}{d_n} \prod_{k=0}^{i-j-1} a_{n-(j+k)}, & i > j \end{cases} \tag{2.4}$$

where d_i $(i = 0,1, \dots, n)$ *represents the determinant leading principal submatrix of order* $i \times i$ *of the T-Sequence -Sylvester-Kac matrix.*

Proof:

$$K_{a,n}^{*}{}^{-1} = \frac{1}{\det(K_{a,n}^{*})} adj(K_{a,n}^{*}).$$

Since $adj(K_{a,n}^{*}) = K^t$ where K^t is a cofactor matrix of $K_{a,n}^{*}$, so by finding the entries of the cofactor matrix $K_{a,n}^{*}$ will get the inverse of the matrix $K_{a,n}^{*}$. For $i,j \neq \{1,n\}$,, the i −th row contains $[0, \ldots, a_{n+1-i}, t, a_i, \ldots, 0]$, while the j −th column contains $[0, \ldots, a_{j-1}, t, a_{n-j}, \ldots, 0]^t$. Therefore, to find the entries of the cofactor matrix $K_{a,n}^{*}$, we will divide by 3 cases, namely:

1. $i = j$

For $i = j$ represents the main diagonal entry of the inverse matrix $K_{a,n}^{*}$, so $K_{ij} = K_{ii}$. K_{ii} declares the determinant of the matrix $K_{a,n}^{*}$ by first removing the $i - th$ column. For $i = 1$ it is obtained that

$$\begin{aligned} K_{11} &= (-1)^{1+1} M_{11} \\ &= d_{n-1}. \end{aligned}$$

And for $i = n$ we get that

$$\begin{aligned} K_{nn} &= (-1)^{n+n} M_{nn} \\ &= d_{n-1}. \end{aligned}$$

As for $i,j \neq \{1,n\}$,then the row containing $[0, \ldots, a_{n+1-i}, t, a_i, \ldots, 0]$, and the column containing $[0, \ldots, a_{i-1}, t, a_{n-i}, \ldots, 0]^t$ are removed, so that its obtained that

$$K_{ii} = (-1)^{i+i} M_{ii}$$

$$K_{ii} = \begin{vmatrix} A & O \\ O & B \end{vmatrix},$$

with

$$A = \begin{bmatrix} t & a_1 & & & \\ a_{n-1} & t & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n+3-i} & t & a_{i-2} \\ & & & a_{n+2-i} & t \end{bmatrix}$$

$$B = \begin{bmatrix} t & a_{i+1} & & & \\ a_{n-(1+i)} & t & a_{i+2} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-2} & t & a_1 \\ & & & a_{n-1} & t \end{bmatrix}$$

so that

$$K_{ii} = \det(A)\det(B).$$

Note that $\det(A) = d_{i-1}$, and

$$\det(B) = \left|\hat{I}_{i-1} B \hat{I}_{i-1}\right| \\ = d_{n-i}.$$

Therefore, $K_{ii} = d_{i-1}d_{n-i}$. Note that for $i = \{1, n\}$ then the equation $K_{ii} = d_{i-1}d_{n-i}$ is also satisfied because $d_0 = 0$. So,

$$K_{ij} = d_{i-1}d_{n-i}\,, \;\; i = j.$$

2. $i < j$

Since $i < j$ on $adj(K^*_{a,n})$ then $i > j$ on K_{ij}. Therefore, by removing the $i - th$ row containing $[0, \dots, a_{n+1-i}, t, a_i, \dots, 0\,]$, and deleting the $j - th$ column containing $\left[0, \dots, a_{j-1}, t, a_{n-j}, \dots, 0\,\right]^t$, then we get

$$K_{ij} = (-1)^{i+j}\begin{vmatrix} A & O & O \\ D & B & O \\ O & E & C \end{vmatrix},$$

with

$$A = \begin{bmatrix} t & a_1 & & & \\ a_{n-1} & t & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n+3-j} & t & a_{j-2} \\ & & & a_{n+2-j} & t \end{bmatrix},$$

$$B = \begin{bmatrix} a_j & & & & \\ t & a_{j+1} & & & \\ a_{n-(i+j)} & \ddots & \ddots & & \\ & \ddots & t & a_{i-2} & \\ & & a_{n+2-i} & t & a_{i-1} \end{bmatrix},$$

$$C = \begin{bmatrix} t & a_{i+1} & & & \\ a_{n-(1+i)} & t & a_{i+2} & & \\ & \ddots & \ddots & \ddots & \\ & & a_2 & t & a_{n-1} \\ & & & a_1 & t \end{bmatrix},$$

$$D = \begin{bmatrix} O & a_{n+1-j} \\ O & O \end{bmatrix},$$

$$E = \begin{bmatrix} O & a_{n-i} \\ O & O \end{bmatrix},$$

so that by using the determinant method for the block matrix, we get

$$K_{ij} = (-1)^{i+j}\det(A)\det(B)\det(C).$$

Since $\det(A) = d_{j-1}$, $\det(B) = a_j a_{j+1} \dots a_{i-1} = \prod_{k=0}^{i-1-j} a_{j+k}$, and

$$\det(C) = \left|\hat{I}_{n-i} C \hat{I}_{n-i}\right|$$

$$= d_{n-i}$$

Therefore, $K_{ij} = (-1)^{i+j} d_{j-1} \prod_{k=0}^{i-j-1} a_{j+k}\, d_{n-i}$ for $i > j$, so on

$$\left(adj(K_{a,n}^*)\right)_{ij} = (-1)^{i+j} d_{i-1} d_{n-j} \prod_{k=0}^{j-i-1} a_{i+k}, \text{ for } i < j.$$

3. $i > j$

Since $i > j$ on $adj(K_{a,n}^*)$ then $i < j$ on K_{ij}. Therefore, by removing the $i - th$ row containing $[0, \dots, a_{n+1-i}, t, a_i, \dots, 0\,]$, and deleting the $j - th$ column containing $\left[0, \dots, a_{j-1}, t, a_{n-j}, \dots, 0\,\right]^t$, then we get

$$K_{ij} = (-1)^{i+j} \begin{vmatrix} A & D & O \\ O & B & E \\ O & O & C \end{vmatrix},$$

with

$$A = \begin{bmatrix} t & a_1 & & & \\ a_{n-1} & t & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n+3-i} & t & a_{i-2} \\ & & & a_{n+2-i} & t \end{bmatrix},$$
$$B = \begin{bmatrix} a_{n-i} & t & a_{i+1} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n+3-j} & t & a_{j-2} \\ & & & a_{n+2-j} & t \\ & & & & a_{n+1-j} \end{bmatrix},$$
$$C = \begin{bmatrix} t & a_{j+1} & & & \\ a_{n-(j+1)} & t & a_{j+2} & & \\ & \ddots & \ddots & \ddots & \\ & & a_2 & t & a_{n-1} \\ & & & a_1 & t \end{bmatrix},$$
$$D = \begin{bmatrix} O & O \\ a_{i-1} & O \end{bmatrix},$$
$$E = \begin{bmatrix} O & O \\ a_j & O \end{bmatrix},$$

so that by using the determinant method for the block matrix, we get

$$K_{ij} = (-1)^{i+j} \det(A) \det(B) \det(C).$$

Since $\det(A) = d_{i-1}, \det(B) = a_{n-i} a_{n-(i+1)} \dots a_{n-(j-1)} = \prod_{k=0}^{j-i-1} a_{n-(i+k)}$, and

$$\det(C) = \left|\hat{I}_{n-i} C \hat{I}_{n-i}\right|$$
$$= d_{n-j}.$$

Therefore, $K_{ij} = (-1)^{i+j} d_{i-1} \prod_{k=0}^{i-j-1} a_{n-(i+k)}\, d_{n-j}$ for $i < j$, so on

$$\left(adj\left(K_{a,n}^{*}\right)\right)_{ij} = (-1)^{i+j} d_{j-1} d_{n-i} \prod_{k=0}^{i-j-1} a_{n-(j+k)}, \text{ for } i > j.$$

So, by multiplying $adj(K_{a,n}^{*})$ by d_n then we get the entries of the matrix ${K_{a,n}^{*}}^{-1}$, as expected. ■

3. CONCLUSION

From the main results above, it is found that the determinant of the T-Sequence-Sylvester-Kac matrix can be expressed as

$$K_{a,n}^{*} = \begin{cases} \displaystyle\sum_{i=0}^{\frac{n-1}{2}} (-1)^{i} C_{n-1,i} t^{n-2i}, & n \text{ odd} \\ \displaystyle\sum_{i=0}^{\frac{n}{2}} (-1)^{i} C_{n-1,i} t^{n-2i}, & n \text{ even} \end{cases}$$

where $C_{n-1,i}$ is stated in Theorem 2.1. While the inverse of T-Sequence-Sylvester-Kac matrix is expressed as

$$\left({K_{a,n}^{*}}^{-1}\right)_{ij} = \begin{cases} \dfrac{d_{i-1} d_{n-i}}{d_n}, & i = j \\ (-1)^{i+j} \dfrac{d_{i-1} d_{n-j}}{d_n} \displaystyle\prod_{k=0}^{j-i-1} a_{i+k}, & i < j \\ (-1)^{i+j} \dfrac{d_{j-1} d_{n-i}}{d_n} \displaystyle\prod_{k=0}^{i-j-1} a_{n-(j+k)}, & i > j \end{cases}$$

where d_i represents the determinant leading principal submatrix determinant of the T-Sequence-Sylvester-Kac. The existence of the inverse matrix T-Sequence-Sylvester-Kac is given by Lemma 2.1.

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REFERENCES

[1] Bevilacqua, R., & Bozzo, E., 2019. The Sylvester-Kac Matrix Space. *Linear Algebra Applications*, *430*, 3131-3138.

[2] Chu, W., 2010. Fibonacci Polynomials and Sylvester Determinant of Tridiagonal Matrix. *Applied Mathematics and Computation*, *216*(3), 1018-1023.

[3] Chu, W., 2019. Spectrum and Eigenvectors for a Class of Tridiagonal Matrices. *Linear Algebra and Its Applications*, *584*, 499-516.

[4] Chu, W., & Wang, X., 2008. Eigenvectors of Tridiagonal Matrices of Sylvester Type. *Calcolo*, *45*(4), 217-233.

[5] Fonseca, C. M., & Kılıç, E., 2020. A Short Note on The Determinant of a Sylvester-Kac Type Matrix. *International Journal of Nonlinear Sciences and Numerical Simulation, 21*, 361-362.

[6] Fonseca, C. M., & Kılıç, E., 2021. A New Type of Sylvester–Kac Matrix and Its Spectrum. *Linear and Multilinear Algebra, 69*, 1072-1082.

[7] Jiang, Z., & Zheng, Y. L., 2022. Characteristic Polynomial, Determinant and Inverse of a Fibonacci-Sylvester-Kac Matrix. *Special Matrices, 10*, 40-46.

[8] Sylvester, J., 1854. Théoreme Sur Les Déterminants. *Nouvelles Annales De Mathématiques, 13*, 305.