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Hopf Bifurcation in a Modified Leslie-Gower Two Preys One Predator Model and Holling Type II Functional Response with Harvesting and Time-Delay

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Abstract

In this paper, a modified Leslie-Gower two preys one predator model and Holling type II functional response with harvesting and time-delay were discussed. Model analysis is carried out by determining fixed points, then analyzing the stability of the fixed points and discussing the existence of the Hopf bifurcation. In some conditions that occur in nature indicate the occurrence of hunting of prey and predator species by humans. Therefore, this model is modified by adding the assumption that prey and predators are being harvested. Another modification given to the model is the use of time delays.The delay time term is for taking into account the case that the members of the predator species need time from birth to predation for being active predators. The first case is a model without time delay, it is obtained that 3 fixed points are unstable and 7 fixed points are stable. One of them is the interior fixed point tested with the Routh-Hurwitz criteria. The second case is a model with a delay time, the critical delay value is obained. Hopf bifurcation occurs when the delay time value is equal to the critical delay value and also fulfills the transversality condition. Observations on the model simulation are carried out by varying the value of the delay time. When the Hopf bifurcation occurs, the graph on the solution plane shows a constant oscillatory movement. If the value of the delay time given is less than the critical value of the delay, the controlled system solution goes to a balanced state. Then when the delay time value is greater than the critical delay value, the system solution continues to fluctuate causing an unstable system condition.

Keywords: Hopf bifurcation; Leslie-Gower; two preys; Holling type II; harvesting; time delay.

1. INTRODUCTION AND PRELIMINARIES

One example of system which is the approach to the physical phenomenon is the predator-prey system [1]. The research of the interaction of predator-prey will be done by analyzing the mathematical model of predator-prey system.

The predator-prey model were first introduced in 1925 by Lotka and Volterra in 1926 [2,3], so this model is also called the Lotka-Volterra model [4]. This simple model has been modified in many ways since its original formulation in the 1920s. In particular, Leslie and Gower [5, 6] improved the realism of the Lotka-Volterra model by introducing the predator-prey model where environmental carrying capacity for the predator population is proportional to the number of prey population. Then in 1950 Holling introduced the functional response. The functional response in ecology i.e. the amount of food is eaten by the predator population as the function of the density of food [7]. In this case the functional response is divided into three kinds i.e. type I functional response, type II functional response and type III functional response. Then to build the model which is more realistic, Ali, et al [8] consider time delay. Delay effect here is to realize the situation that the predator population needs time from birth to predation. This means that a newborn predator species need time to grow old enough to predate alone and to be represented in the predation function mathematically.

The Leslie Gower predator-prey model and Holling type 2 functional response with time delay were introduced by Nindjin et al [9] after that Ma [10] added a time delay to the previous model with the assumption that the growth of the prey population depended on the previous population size. Aziz and Daher [11] studied global stability of the coexisting interior equilibrium in modified Leslie-Gower Predator-prey and Holling Type II Functional Response. Pan-Ping & Yong [12] analyzed spatiotemporal dynamics of a predator–prey model obtain rich patterns, including spotted, black-eye, and labyrinthine patterns with choosing appropriate parameter values. Tapan & Charugopala [13] proposed a delayed Holling–Tanner predator–prey model with ratio-dependent functional response. The local stability, Hopf-bifurcation, qualitative behaviour of the singularity with blow up transformation and global stability are discussed. Zhiqing and Hongwei [14] investigated the global stability of equilibrium of Holling–Tanner model with ratio-dependence by constructing Lyapunov function and discuss the existence limit cycle of model linear. Malay and Santo [15] considered a modified spatiotemporal ecological system originating from the temporal Holling–Tanner model with diffusion. The original ODE system is studied for the stability of coexisting homogeneous steady-states. The modified PDE system is investigated in detail with both numerical and analytical approaches. Turing and non-Turing patterns of fixed values and Hopf bifurcation are discussed. Zizhen Zhang [16] analyzed Hopf bifurcation analysis for a two prey one predator system with time delay and show the hybrid controller is efficient in controlling Hopf bifurcation. Xinzhi, et al [17] studied the existence of invasion waves of a diffusive predator–prey model with two preys and one predator. The existence of traveling semi-fronts connecting invasion-free equilibrium is obtained by Schauder's fixed-point theorem The existence of traveling front is got by rescaling method and limit arguments. Gakkhar, S and Kamel Naji, R [18] investigated the dynamical behavior and chaos of a realistic three species food chain model considering predator to prey ratio-dependence for the interaction together with type II functional response. The bifurcation diagrams, Lyapunov exponents and dimensions are discussed. Xiaoliang and Wen [19] concerned a three-species Lotka-Volterra food chain system with multiple delays. The direction and the stability of bifurcating periodic solutions are determined by the normal form theory and the center manifold theorem. Nilesh, et al [20] discussed modeling the plankton–fish. Interaction between the prey and an intermediate predator follows the Monod–Haldane functional response, while that between the top predator and its prey depends on Beddington–DeAngelis-type functional response. we study the Hopf and transcritical bifurcations scenarios with respect to inhibitory effect of phytoplankton against zooplankton and death rate of fish population for the non-delayed system. Jia-Fang [21] studied the direction of Hopf bifurcation and explicit algorithm is given by applying the normal form theory and the center manifold reduction. Zhiqi and Xia [22] discussed a predator–prey model with modified Holling–Tanner functional response and time delay. The local stability and global stability with Lyapunov theorem is investigated.

The model discussed in this study is a modification of the model formulated by Ali, et al [8] with the assumption that in some conditions that occur in nature indicate the occurrence of hunting by humans. To control the level of predation so as not to cause extinction of the prey and predator species, a constant harvesting treatment is given to the prey and predator populations on a regular basis. In general, harvesting activities are carried out on individuals who have reached a certain age which are considered mature or ripe for harvesting. According to Idels and Wong [23] constant harvesting does not increase or decrease every year. In this research, it is assumed that harvesting can produce results that do not cause the species to become extinct. Based on the author's literature review, this modified model has never been studied before. Model analysis is carried out by determining the fixed point, then a stability analysis is performed from the fixed point without time delay including interior points tested with Routh Hurwitz criteria and time delay interior fixed points by applying Hopf bifurcation theory which is continued with model simulations carried out by varying the value of delay time.

2. METHOD

The stages of the research carried out are as follows.

1. Journal Review
   In general, the content of the article is about the analysis of the stability of the Leslie Gower equation and the Holling type II response function with harvesting and delay time.
2. Model Building
   The modified Leslie Gower model and the Holling type II response function which will be studied in this study were obtained from the development of previous research models.
3. Fixed Point determination stage
   The fixed point is obtained by making the rate of change of predator and prey with respect to time equal to zero.
4. Fixed Point Stability Analysis Stage
   Fixed point stability without time delay is obtained by a linear approach. The interior points are substituted into the Jacobi matrix so that the eigenvalues can be analyzed with Routh Hurwitz to determine stability. A fixed point with a time delay requires an approach in complex space to analyze the Hopf bifurcation.
5. Stage of Determination of Transverse Conditions
   The transverse condition is determined to prove that a Hopf bifurcation occurs at the inner equilibrium point.
6. Interior Point Stability Simulation Stage
   Simulations are carried out for each parameter according to the conditions. The simulation results were analyzed, in order to obtain an overview of the influence of harvesting and delay time on predators and prey.

3. MATHEMATICAL MODEL

A modified Leslie-Gower two preys one predator model and Holling type II functional response and harvesting consists of two different prey density at time $t\ \left(x_1(t)\ dan\ x_2(t)\right)$ and the predator population at time $t\ \left(x_3(t)\right)$.

The prey population density depends on several factors, including the growth rate of each prey and predator population, carrying capacity for the prey population, the maximum consumption rate of the predator population and the saturation rate of the predation.

The predator population density depends on several factors, including the growth rate of the predator population, the maximum consumption rate of the predator population and the saturation rate of the predation.

In the formation of this model, there are several assumptions. The assumptions is used in the modified Leslie-Gower two preys one predator model and Holling type II functional response and harvesting are (1) The predator can eat both preys, meantime there is no interaction between x_1 and x_2 (2) The growth rate of the prey population has a logistic growth pattern. (3) In the interaction between prey and predator, the prey responds the presence of the predator, such that the predator takes time to catch the prey (predator following the Holling type II functional response). (4) There is the harvesting in the prey and predator population after the prey and predator population density reaches a threshold harvesting. A modified Leslie-Gower two preys one predator model and Holling type II functional response and harvesting can be expressed as.

(1) $\frac{dx_1}{dt} = r_1x_1\left(1-\frac{x_1}{K_1}\right)-\frac{a_1x_1x_3}{n_1+x_1}-r_1Fx_1,$

(2) $\frac{dx_2}{dt} = r_2x_2\left(1-\frac{x_2}{K_2}\right)-\frac{a_2x_2x_3}{n_2+x_2}-r_2Fx_2,$

(3) $\frac{dx_3}{dt} = s_1x_3\left(1-\frac{q_1x_3}{K_1}\right)+s_2x_3\left(1-\frac{q_2x_3}{K_2}\right)-s_1Fx_3-s_2Fx_3.$ (1)

with $x_1(0)>0, x_2(0)>0$ and $x_3(0)>0$.

$x_1 = K_1x, x_2 = K_2y, a_1x_3 = K_1z \Leftrightarrow x_3 = \frac{K_1z}{a_1}, a_2x_3 = K_2z \Leftrightarrow x_3 = \frac{K_2z}{a_2}, r_1t = T \Leftrightarrow t = \frac{T}{r_1}, r_2t = T \Leftrightarrow t = \frac{T}{r_2}$ $\alpha_1 = \frac{1}{r_1}, \alpha_2 = \frac{1}{r_2}$ $\beta_1 = \frac{q_1}{a_1}, \beta_2 = \frac{q_2}{a_2}$ $m_1 = \frac{n_1}{K_1} \Leftrightarrow n_1 = m_1K_1, m_2 = \frac{n_2}{K_2} \Leftrightarrow n_2 = m_2K_2, \rho_1 = \frac{s_1}{r_1} \Leftrightarrow s_1 = \rho_1r_1, \rho_2 = \frac{s_2}{r_2} \Leftrightarrow s_2 = \rho_2r_2$. In the equation system (1), the parameter $x = \frac{x_1}{K_1}$ and $y = \frac{x_2}{K_2}$ describes that the prey population density with the influence of the surroundings, the parameter $z = \frac{a_1x_3}{K_1} = \frac{a_2x_3}{K_2}$ shows that the predator population density with the influence of the surroundings which is interacting with the prey population, $T = r_1t = r_2t$ shows the time t, $\alpha_1 = \frac{1}{r_1}$ and $\alpha_2 = \frac{1}{r_2}$ describes that the decrease number of the prey population that caused the interaction of the prey and predator population, $\beta_1 = \frac{q_1}{a_1}, \beta_2 = \frac{q_2}{a_2}$ shows that the decrease number of the predator population that caused the interaction between one predator and the other predator population, $m_1 = \frac{n_1}{K_1}, m_2 = \frac{n_2}{K_2}$ describes that the saturation rate of the predation with the influence of the surroundings and $\rho_1 = \frac{s_1}{r_1}, \rho_2 = \frac{s_2}{r_2}$ describes that the growth rate of the predator population. So, system (1) is equivalent with the following system.

(1) $\frac{dx}{dT} = x\left(1-x-\frac{\alpha_1z}{m_1+x}-F\right),$

(2) $\frac{dy}{dT} = y\left(1-y-\frac{\alpha_2z}{m_2+y}-F\right),$

(3) $\frac{dz}{dT} = z(\rho_1(1-\beta_1z)+\rho_2(1-\beta_2z)-F)$

(2)

with $x(0)>0, y(0)>0$ and $z(0)>0$.

Then by giving the discrete time-delay in the growth rate of the predator population, the equation models become

(1) $\frac{dx(T)}{dT} = x(T)\left(1-x(T)-\frac{\alpha_1z(T-\tau)}{m_1+x(T-\tau)}-F\right)$

(2) $\frac{dy(T)}{dT} = y(T)\left(1 - y(T) - \frac{\alpha_2 z(T-\tau)}{m_2 + y(T-\tau)} - F\right)$

(3) $\frac{dz(T)}{dT} = z(T)\big(\rho_1\big(1 - \beta_1 z(T-\tau)\big) + \rho_2\big(1 - \beta_2 z(T-\tau)\big) - F\big)$
(3)

with $x(0) > 0, y(0) > 0$ and $z(0) > 0$.

4. MAIN RESULTS

4.1 EQUILIBRIA

System (3) realizes the equilibrium point when $\frac{dx}{dT} = 0, \frac{dy}{dT} = 0$ and $\frac{dz}{dT} = 0$, such that system (3) can be written as.

(a) $\frac{dx}{dT} = x\left(1 - x - \frac{\alpha_1 z}{m_1 + x} - F\right) = 0,$

(b) $\frac{dy}{dT} = y\left(1 - y - \frac{\alpha_2 z}{m_2 + y} - F\right) = 0,$

(c) $\frac{dz}{dT} = z(\rho_1(1 - \beta_1 z) + \rho_2(1 - \beta_2 z) - F) = 0$ (4)

From the equation c in system (4) we obtain $z = 0$ or $z = \frac{\rho_1 + \rho_2 - F}{\rho_1\beta_1 + \rho_2\beta_2}$ with $\rho_1 + \rho_2 > F$.

For the case $z = 0$, substituting $z = 0$ into the equation (a) and (b) in system (4), such that we get $y = 0 \vee y = 1$ and $x = 0 \vee x = 1$. Hence, we obtain the equilibrium point $E_0(0,0,0), E_1(0,1,0), E_2(1,0,1), E_3(1,1,0)$. For the case $z = \frac{\rho_1 + \rho_2 - F}{\rho_1\beta_1 + \rho_2\beta_2}$, substituting $z = \frac{\rho_1 + \rho_2 - F}{\rho_1\beta_1 + \rho_2\beta_2}$ into the equation (a) and (b) in system (4), then we obtain $x = 0 \vee x = -\frac{1}{2}(F - 1 + m_1 \pm A)$ and $y = 0 \vee y = -\frac{1}{2}(F - 1 + m_1 \pm B)$ with $A = \sqrt{\frac{(F - (m_1 + 1))^2(\rho_1\beta_1 + \rho_2\beta_2) - 4\alpha_1(\rho_1 + \rho_2 - F)}{\rho_1\beta_1 + \rho_2\beta_2}}$ and $B = \sqrt{\frac{(F - (m_2 + 1))^2(\rho_1\beta_1 + \rho_2\beta_2) - 4\alpha_2(\rho_1 + \rho_2 - F)}{\rho_1\beta_1 + \rho_2\beta_2}}$ where $F - 1 + m_1 \pm A < 0$ and $F - 1 + m_1 \pm B < 0$. So we have the equilibrium point $E_4\left(0,0,\frac{\rho_1 + \rho_2 - F}{\rho_1\beta_1 + \rho_2\beta_2}\right), E_5\left(-\frac{1}{2}(F - 1 + m_1 + A), 0, \frac{\rho_1 + \rho_2 - F}{\rho_1\beta_1 + \rho_2\beta_2}\right), E_6\left(0, -\frac{1}{2}(F - 1 + m_1 + B), \frac{\rho_1 + \rho_2 - F}{\rho_1\beta_1 + \rho_2\beta_2}\right), E_7\left(-\frac{1}{2}(F - 1 + m_1 + A), -\frac{1}{2}(F - 1 + m_1 + B), \frac{\rho_1 + \rho_2 - F}{\rho_1\beta_1 + \rho_2\beta_2}\right), E_8\left(-\frac{1}{2}(F - 1 + m_1 - A), 0, \frac{\rho_1 + \rho_2 - F}{\rho_1\beta_1 + \rho_2\beta_2}\right), E_9\left(0, -\frac{1}{2}(F - 1 + m_1 - B), \frac{\rho_1 + \rho_2 - F}{\rho_1\beta_1 + \rho_2\beta_2}\right), E_{10}\left(-\frac{1}{2}(F - 1 + m_1 - A), -\frac{1}{2}(F - 1 + m_1 - B), \frac{\rho_1 + \rho_2 - F}{\rho_1\beta_1 + \rho_2\beta_2}\right)$.

4.2 STABILITY OF $\hat{E}$ WITHOUT TIME DELAY

The general Jacobian matrix of Eqs. (4) is given by

$$J = \begin{pmatrix} 1 - 2x - \frac{\alpha_1 z}{m_1 + x} + \frac{\alpha_1 x z}{(m_1 + x)^2} - F & 0 & \frac{-\alpha_1 x}{m_1 + x} \\ 0 & 1 - 2y - \frac{\alpha_2 z}{m_2 + y} + \frac{\alpha_2 y z}{(m_2 + y)^2} - F & \frac{-\alpha_2 y}{m_2 + y} \\ 0 & 0 & \rho_1(1 - \beta_1 z) - \rho_1 z\beta_1 + \rho_2(1 - \beta_2 z) - \rho_2 z\beta_2 - F \end{pmatrix}$$

At E_0, the Jacobian matrix is $J(0,0,0) = \begin{pmatrix} 1 - F & 0 & 0 \\ 0 & 1 - F & 0 \\ 0 & 0 & \rho_1 + \rho_2 - F \end{pmatrix}$. We get $\lambda_{1,2} = 1 - F \vee \lambda_3 = \rho_1 + \rho_2 - F$. Because $1 > F$ and $\rho_1 + \rho_2 > F$, E_0 is an unstable saddle point.

At E_1, the Jacobian matrix is $J(0,1,0)=\begin{pmatrix}1-F & 0 & 0\\ 0 & -1-F & -\frac{\alpha_2}{m_2+1}\\ 0 & 0 & \rho_1+\rho_2-F\end{pmatrix}$. We get $\lambda_{1,2}=1-F \vee \lambda_3=\rho_1+\rho_2-F$. Hence, E_1 is an unstable saddle point.

At E_2, the Jacobian matrix is $J(1,0,1)=$

$$\begin{pmatrix}-\frac{(m_1+1)^2(F+1)+\alpha_1 m_1}{(m_1+1)^2} & 0 & -\frac{\alpha_1}{m_1+1}\\ 0 & -\frac{m_2(F-1)+\alpha_2}{m_2} & 0\\ 0 & 0 & \rho_1+\rho_2-F-2(\rho_1\beta_1+\rho_2\beta_2)\end{pmatrix}$$

. We get $\lambda_1=-\frac{(m_1+1)^2(F+1)+\alpha_{1m_1}}{(m_1+1)^2} \vee \lambda_2=-\frac{m_2(F-1)+\alpha_2}{m_2} \vee \lambda_3=\rho_1+\rho_2-F-2(\rho_1\beta_1+\rho_2\beta_2)$. E_2 stable if $m_2(F-1)+\alpha_2>0$ and $\rho_1+\rho_2-F<2(\rho_1\beta_1+\rho_2\beta_2)$.

At E_3, the Jacobian matrix is $J(1,1,0)=\begin{pmatrix}-1-F & 0 & -\frac{\alpha_1}{m_1+1}\\ 0 & -1-F & -\frac{\alpha_2}{m_2+1}\\ 0 & 0 & \rho_1+\rho_2-F\end{pmatrix}$. We get $\lambda_{1,2}=1-F \vee \lambda_3=\rho_1+\rho_2-F$. Hence, E_3 is an unstable saddle point.

At E_4, the Jacobian matrix is $J(1,1,0)=$

$$\begin{pmatrix}-\frac{\rho_1\beta_1 m_1(-1+F)+\rho_2\beta_2 m_1(-1+F)+\alpha_1(\rho_1+\rho_2-F)}{(\rho_1\beta_1+\rho_2\beta_2)m_1} & 0 & 0\\ 0 & -\frac{\rho_1\beta_1 m_2(-1+F)+\rho_2\beta_2 m_2(-1+F)+\alpha_2(\rho_1+\rho_2-F)}{(\rho_1\beta_1+\rho_2\beta_2)m_2} & 0\\ 0 & 0 & -\rho_1-\rho_2+F\end{pmatrix}$$

. We get $\lambda_1=-\frac{\rho_1\beta_1 m_1(-1+F)+\rho_2\beta_2 m_1(-1+F)+\alpha_1(\rho_1+\rho_2-F)}{(\rho_1\beta_1+\rho_2\beta_2)m_1} \vee \lambda_2=-\frac{\rho_1\beta_1 m_2(-1+F)+\rho_2\beta_2 m_2(-1+F)+\alpha_2(\rho_1+\rho_2-F)}{(\rho_1\beta_1+\rho_2\beta_2)m_2} \vee \lambda_3=-\rho_1-\rho_2+F$. Hence, E_4 stable if $\rho_1\beta_1 m_1(-1+F)+\rho_2\beta_2 m_1(-1+F)+\alpha_1(\rho_1+\rho_2-F)>0$ and $\rho_1\beta_1 m_2(-1+F)+\rho_2\beta_2 m_2(-1+F)+\alpha_2(\rho_1+\rho_2-F)>0$.

At E_5, the Jacobian matrix is $J\left(-\frac{1}{2}(F-1+m_1+A),0,\frac{\rho_1+\rho_2-F}{\rho_1\beta_1+\rho_2\beta_2}\right)=$

$$\begin{pmatrix}\frac{(A+\alpha_1)(-m_1+F-1+A)^2(\rho_1\beta_1+\rho_2\beta_2)+4\alpha_1 m_1(-\rho_1-\rho_2+F)}{(\rho_1\beta_1+\rho_2\beta_2)(-m_1+F-1+A)^2} & 0 & -\frac{\alpha_1(F-1+m_1+A)}{-m_1+F-1+A}\\ 0 & -\frac{\alpha_2(\rho_1+\rho_2-F)+m_2(F-1)(\rho_1\beta_1+\rho_2\beta_2)}{(\rho_1\beta_1+\rho_2\beta_2)m_2} & 0\\ 0 & 0 & -\rho_1-\rho_2+F\end{pmatrix}.$$

We get $\lambda_1=\frac{(A+\alpha_1)(-m_1+F-1+A)^2(\rho_1\beta_1+\rho_2\beta_2)+4\alpha_1 m_1(-\rho_1-\rho_2+F)}{(\rho_1\beta_1+\rho_2\beta_2)(-m_1+F-1+A)^2} \vee \lambda_2=-\frac{\alpha_2(\rho_1+\rho_2-F)+m_2(F-1)(\rho_1\beta_1+\rho_2\beta_2)}{(\rho_1\beta_1+\rho_2\beta_2)m_2} \vee \lambda_3=-\rho_1-\rho_2+F$. Hence, E_5 stable if $(A+\alpha_1)(-m_1+F-1+A)^2(\rho_1\beta_1+\rho_2\beta_2)+4\alpha_1 m_1(-\rho_1-\rho_2+F)<0$ dan $\alpha_2(\rho_1+\rho_2-F)+m_2(F-1)(\rho_1\beta_1+\rho_2\beta_2)>0$.

At E_6, the Jacobian matrix is $J\left(0,-\frac{1}{2}(F-1+m_1+B),\frac{\rho_1+\rho_2-F}{\rho_1\beta_1+\rho_2\beta_2}\right)=$

$$\begin{pmatrix}-\frac{\alpha_1(\rho_1+\rho_2-F)+m_1(F-1)(\rho_1\beta_1+\rho_2\beta_2)}{(\rho_1\beta_1+\rho_2\beta_2)m_1} & 0 & 0\\ 0 & \frac{(B+m_1)(-2m_2+F-1+m_1+B)^2(\rho_1\beta_1+\rho_2\beta_2)+4\alpha_2 m_2(F-\rho_2-\rho_1)}{(\rho_1\beta_1+\rho_2\beta_2)(-2m_2+F-1+m_1+B)^2} & -\frac{\alpha_2(F-1+m_1+B)}{-2m_2+F-1+m_1+B}\\ 0 & 0 & -\rho_1-\rho_2+F\end{pmatrix}$$

. We get $\lambda_1=-\frac{\alpha_1(\rho_1+\rho_2-F)+m_1(F-1)(\rho_1\beta_1+\rho_2\beta_2)}{(\rho_1\beta_1+\rho_2\beta_2)m_1} \vee \lambda_2=\frac{(B+m_1)(-2m_2+F-1+m_1+B)^2(\rho_1\beta_1+\rho_2\beta_2)+4\alpha_2 m_2(F-\rho_2-\rho_1)}{(\rho_1\beta_1+\rho_2\beta_2)(-2m_2+F-1+m_1+B)^2} \vee \lambda_3=-\frac{\alpha_2(F-1+m_1+B)}{-2m_2+F-1+m_1+B}$. Hence, E_6

stable if $\alpha_1(\rho_1+\rho_2-F)+m_1(F-1)(\rho_1\beta_1+\rho_2\beta_2)>0$ and $(B+m_1)(-2m_2+F-1+m_1+B)^2(\rho_1\beta_1+\rho_2\beta_2)+4\alpha_2 m_2(F-\rho_2-\rho_1)<0$.

At E_8, the Jacobian matrix is $J\left(-\frac{1}{2}(F-1+m_1-A),0,\frac{\rho_1+\rho_2-F}{\rho_1\beta_1+\rho_2\beta_2}\right)=$

$$\begin{pmatrix} -\frac{(A-m_1)(m_1-F+1+A)^2(\rho_1\beta_1+\rho_2\beta_2)-4\alpha_1 m_1(-\rho_1-\rho_2+F)}{(\rho_1\beta_1+\rho_2\beta_2)(m_1-F+1+A)^2} & 0 & -\frac{\alpha_1(-F+1-m_1+A)}{m_1-F+1+A} \\ 0 & -\frac{-m_2(\rho_1\beta_1+\rho_2\beta_2)+\alpha_2(\rho_1+\rho_2-F)+Fm_2(\rho_1\beta_1+\rho_2\beta_2)}{(\rho_1\beta_1+\rho_2\beta_2)m_2} & 0 \\ 0 & 0 & -\rho_1-\rho_2+F \end{pmatrix}.$$

We get $\lambda_1=-\frac{(A-m_1)(m_1-F+1+A)^2(\rho_1\beta_1+\rho_2\beta_2)-4\alpha_1 m_1(-\rho_1-\rho_2+F)}{(\rho_1\beta_1+\rho_2\beta_2)(m_1-F+1+A)^2}\vee\lambda_2=-\frac{-m_2(\rho_1\beta_1+\rho_2\beta_2)+\alpha_2(\rho_1+\rho_2-F)+Fm_2(\rho_1\beta_1+\rho_2\beta_2)}{(\rho_1\beta_1+\rho_2\beta_2)m_2}\vee\lambda_3=-\rho_1-\rho_2+F$. Hence, E_8 stable if $(A-m_1)(m_1-F+1+A)^2(\rho_1\beta_1+\rho_2\beta_2)>4\alpha_1 m_1(-\rho_1-\rho_2+F)$ and $\alpha_2(\rho_1+\rho_2-F)+Fm_2(\rho_1\beta_1+\rho_2\beta_2)>m_2(\rho_1\beta_1+\rho_2\beta_2)$.

At E_9 the Jacobian matrix is $J\left(0,-\frac{1}{2}(F-1+m_1-B),\frac{\rho_1+\rho_2-F}{\rho_1\beta_1+\rho_2\beta_2}\right)=$

$$\begin{pmatrix} -\frac{(B-m_1)(\rho_1\beta_1+\rho_2\beta_2)(2m_2-F+1-m_1+B)^2-4\alpha_2 m_2(-\rho_1-\rho_2+F)}{(\rho_1\beta_1+\rho_2\beta_2)(2m_2-F+1-m_1+B)^2} & 0 & -\frac{\alpha_2(-F+1-m_1+B)}{2m_2-m_1-F+1+B} \\ 0 & -\frac{-m_1(\rho_1\beta_1+\rho_2\beta_2)+\alpha_1(\rho_1+\rho_2-F)+Fm_1(\rho_1\beta_1+\rho_2\beta_2)}{(\rho_1\beta_1+\rho_2\beta_2)m_1} & 0 \\ 0 & 0 & -\rho_1-\rho_2+F \end{pmatrix}.$$

We get $\lambda_1=-\frac{(B-m_1)(\rho_1\beta_1+\rho_2\beta_2)(2m_2-F+1-m_1+B)^2-4\alpha_2 m_2(-\rho_1-\rho_2+F)}{(\rho_1\beta_1+\rho_2\beta_2)(2m_2-F+1-m_1+B)^2}\vee\lambda_2=-\frac{-m_1(\rho_1\beta_1+\rho_2\beta_2)+\alpha_1(\rho_1+\rho_2-F)+Fm_1(\rho_1\beta_1+\rho_2\beta_2)}{(\rho_1\beta_1+\rho_2\beta_2)m_1}\vee\lambda_3=-\rho_1-\rho_2+F$. Hence, E_9 stable if $(B-m_1)(\rho_1\beta_1+\rho_2\beta_2)(2m_2-F+1-m_1+B)^2>4\alpha_2 m_2(-\rho_1-\rho_2+F)$ and $\alpha_1(\rho_1+\rho_2-F)+Fm_1(\rho_1\beta_1+\rho_2\beta_2)>m_1(\rho_1\beta_1+\rho_2\beta_2)$.

At E_{10} the Jacobian matrix is $J\left(-\frac{1}{2}(F-1+m_1-A),-\frac{1}{2}(F-1+m_1-B),\frac{\rho_1+\rho_2-F}{\rho_1\beta_1+\rho_2\beta_2}\right)=$

$$\begin{pmatrix} -\frac{(A-m_1)(\rho_1\beta_1+\rho_2\beta_2)(m_1-F+1+A)^2-4\alpha_1 m_1(-\rho_1-\rho_2+F)}{(\rho_1\beta_1+\rho_2\beta_2)(m_1-F+1+A)^2} & 0 & -\frac{\alpha_1(-F+1-m_1+A)}{m_1-F+1+A} \\ 0 & -\frac{(B-m_1)(\rho_1\beta_1+\rho_2\beta_2)(2m_2-F+1-m_1+B)^2-4\alpha_2 m_2(-\rho_1-\rho_2+F)}{(\rho_1\beta_1+\rho_2\beta_2)(2m_2-F+1-m_1+B)^2} & -\frac{\alpha_2(-F+1-m_1+B)}{2m_2-F+1-m_1+B} \\ 0 & 0 & -\rho_1-\rho_2+F \end{pmatrix}.$$

We get $\lambda_1=-\frac{(A-m_1)(\rho_1\beta_1+\rho_2\beta_2)(m_1-F+1+A)^2-4\alpha_1 m_1(-\rho_1-\rho_2+F)}{(\rho_1\beta_1+\rho_2\beta_2)(m_1-F+1+A)^2}\vee\lambda_2=-\frac{(B-m_1)(\rho_1\beta_1+\rho_2\beta_2)(2m_2-F+1-m_1+B)^2-4\alpha_2 m_2(-\rho_1-\rho_2+F)}{(\rho_1\beta_1+\rho_2\beta_2)(2m_2-F+1-m_1+B)^2}\vee\lambda_3=-\rho_1-\rho_2+F$. Hence, E_{10} stable if $(A-m_1)(\rho_1\beta_1+\rho_2\beta_2)(m_1-F+1+A)^2>4\alpha_1 m_1(-\rho_1-\rho_2+F)$ and $(B-m_1)(\rho_1\beta_1+\rho_2\beta_2)(2m_2-F+1-m_1+B)^2>4\alpha_2 m_2(-\rho_1-\rho_2+F)$.

At $\hat{E}$ the Jacobian matrix is $J(\hat{x},\hat{y},\hat{z})=$

$$\begin{pmatrix} 1-2\hat{x}-\frac{\alpha_1\hat{z}}{m_1+\hat{x}}+\frac{\alpha_1\hat{x}\hat{z}}{(m_1+\hat{x})^2}-F & 0 & \frac{-\alpha_1\hat{x}}{m_1+\hat{x}} \\ 0 & 1-2\hat{y}-\frac{\alpha_2\hat{z}}{m_2+\hat{y}}+\frac{\alpha_2\hat{y}\hat{z}}{(m_2+\hat{y})^2}-F & \frac{-\alpha_2\hat{y}}{m_2+\hat{y}} \\ 0 & 0 & \rho_1(1-\beta_1\hat{z})-\rho_1\hat{z}\beta_1+\rho_2(1-\beta_2\hat{z})-\rho_2\hat{z}\beta_2-F \end{pmatrix}.$$

Then find all eigenvalues of a matrix using the characteristic equation as follows.

$$\lambda^3-\psi\lambda^2-\delta\lambda-\eta=0$$

$$\psi=2+\frac{\alpha_2\hat{y}z}{(m_2+\hat{y})^2}+\rho_1+\rho_2-3F-2\rho_1\hat{z}\beta_1-\frac{\alpha_1\hat{z}}{m_1+\hat{x}}-2\rho_2\hat{z}\beta_2-2\hat{y}-\frac{\alpha_2\hat{z}}{m_2+\hat{y}}-2\hat{x}+\frac{\alpha_1\hat{x}\hat{z}}{(m_1+\hat{x})^2}$$

$$\delta=-1-\frac{2\alpha_1\hat{z}F}{m_1+\hat{x}}-4\hat{x}\rho_2\hat{z}\beta_2-4\hat{x}\rho_1\hat{z}\beta_1+\frac{\alpha_1\hat{z}}{m_1+\hat{x}}+\frac{2F\alpha_2\hat{y}\hat{z}}{(m_2+\hat{y})^2}-3F^2-4F\hat{x}+\frac{2\alpha_1\hat{x}\hat{z}F}{(m_1+\hat{x})^2}-\frac{\alpha_1\hat{x}\hat{z}\rho_1}{(m_1+\hat{x})^2}+\frac{2\alpha_1\hat{x}\hat{z}^2\rho_1\beta_1}{(m_1+\hat{x})^2}-\frac{\alpha_1\hat{x}\hat{z}\rho_2}{(m_1+\hat{x})^2}+\frac{\alpha_2\hat{z}}{m_2+\hat{y}}-4F\hat{y}+\frac{2\hat{x}\alpha_2\hat{y}\hat{z}}{(m_2+\hat{y})^2}-4F\rho_2\hat{z}\beta_2-4F\rho_1\hat{z}\beta_1-2\rho_1-2\rho_2+\frac{2\alpha_1\hat{x}\hat{z}^2\rho_2\beta_2}{(m_1+\hat{x})^2}+2\hat{y}\rho_1+2\hat{y}\rho_2+4\rho_1\hat{z}\beta_1+4\rho_2\hat{z}\beta_2-\frac{\alpha_1\hat{x}\hat{z}}{(m_1+x)^2}-\frac{\alpha_2\hat{y}\hat{z}}{(m_2+\hat{y})^2}+\frac{\alpha_1\hat{z}^2\alpha_2\hat{y}}{(m_1+\hat{x})(m_2+\hat{y})^2}+\frac{\alpha_1\hat{x}\hat{z}^2\alpha_2}{(m_1+\hat{x})^2(m_2+\hat{y})}-\frac{2\alpha_1\hat{z}^2\alpha_2}{(m_1+\hat{x})(m_2+\hat{y})}-\frac{2\alpha_1\hat{x}\hat{z}^2\alpha_2\hat{y}}{(m_1+\hat{x})^2(m_2+\hat{y})^2}-\frac{2\alpha_1\hat{z}\hat{y}}{m_1+\hat{x}}+\frac{2\alpha_1\hat{x}\hat{z}\hat{y}}{(m_1+\hat{x})^2}+\frac{\alpha_1\hat{z}\rho_2}{m_1+\hat{x}}+2\hat{y}+\frac{\alpha_1\hat{z}\rho_1}{m_1+\hat{x}}-$$

$$\frac{2\alpha_2\hat{z}^2\rho_1\beta_1}{m_2+\hat{y}}-\frac{2\alpha_2\hat{z}^2\rho_2\beta_2}{m_2+\hat{y}}-\frac{2\hat{x}\alpha_2\hat{z}}{m_2+\hat{y}}-\frac{2F\alpha_2\hat{z}}{m_2+\hat{y}}-\frac{\alpha_2\hat{y}\hat{z}\rho_1}{(m_2+\hat{y})^2}+\frac{2\alpha_2\hat{y}\hat{z}^2\rho_1\beta_1}{(m_2+\hat{y})^2}-\frac{\alpha_2\hat{y}\hat{z}\rho_2}{(m_2+\hat{y})^2}+\frac{2\alpha_2\hat{y}\hat{z}^2\rho_2\beta_2}{(m_2+\hat{y})^2}+2\hat{x}-$$
$$\frac{\alpha_1 z\lambda}{m_1+\hat{x}}+4F-4\hat{x}\hat{y}-\frac{2\alpha_1\hat{z}^2\rho_2\beta_2}{m_1+\hat{x}}+2\hat{x}\rho_1+2\hat{x}\rho_2+\frac{\alpha_2\hat{z}\rho_2}{m_2+\hat{y}}+\frac{\alpha_2\hat{z}\rho_1}{m_2+\hat{y}}-4\hat{y}\rho_1\hat{z}\beta_1-4\hat{y}\rho_2\hat{z}\beta_2+$$
$$2F\rho_1+2F\rho_2$$

$$\eta=-2F^2\rho_1\hat{z}\beta_1+4\hat{x}\rho_1\hat{z}\beta_1-2F^2\rho_2\hat{z}\beta_2-2\rho_2\hat{z}\beta_2-2\rho_1\hat{z}\beta_1-\frac{\alpha_1\hat{z}^2\alpha_2\hat{y}\rho_2}{(m_1+\hat{x})(m_2+\hat{y})^2}-$$
$$\frac{\alpha_1 x\hat{z}^2\alpha_2\rho_1}{(m_1+\hat{x})^2(m_2+\hat{y})}+\frac{2\alpha_1\hat{x}\hat{z}^3\alpha_2\rho_1\beta_1}{(m_1+\hat{x})^2(m_2+\hat{y})}-\frac{\alpha_1\hat{x}\hat{z}^2\alpha_2\rho_2}{(m_1+\hat{x})^2(m_2+\hat{y})}+\frac{2\alpha_1\hat{x}\hat{z}^3\alpha_2\rho_2\beta_2}{(m_1+\hat{x})^2(m_2+\hat{y})}+\frac{\alpha_1 x\hat{z}^2\alpha_2 F}{(m_1+\hat{x})^2(m_2+\hat{y})}+\frac{\alpha_1\hat{z}F\rho_1}{m_1+\hat{x}}-$$
$$\frac{2\alpha_1\hat{z}^2F\rho_1\beta_1}{m_1+\hat{x}}+\frac{\alpha_1\hat{z}F\rho_2}{m_1+\hat{x}}-\frac{\alpha_1\hat{z}\rho_2}{m_1+\hat{x}}-\frac{\alpha_1\hat{z}\rho_1}{m_1+\hat{x}}+\frac{2F\alpha_2\hat{y}\hat{z}^2\rho_2\beta_2}{(m_2+\hat{y})^2}+4\hat{y}\rho_1\hat{z}\beta_1-\frac{\alpha_1\hat{x}\hat{z}F}{(m_1+\hat{x})^2}-\frac{F\alpha_2\hat{y}\hat{z}}{(m_2+\hat{y})^2}+$$
$$4\hat{y}\rho_2\hat{z}\beta_2+\frac{2\alpha_2\hat{z}^2\rho_1\beta_1}{m_2+\hat{y}}+\frac{2\alpha_2\hat{z}^2\rho_2\beta_2}{m_2+\hat{y}}-4F\hat{y}\rho_1\hat{z}\beta_1-4F\hat{y}\rho_2\hat{z}\beta_2-\frac{2\hat{x}\alpha_2\hat{y}\hat{z}\rho_1}{(m_2+\hat{y})^2}+\frac{4\hat{x}\alpha_2\hat{y}\hat{z}^2\rho_1\beta_1}{(m_2+\hat{y})^2}-$$
$$\frac{2\hat{x}\alpha_2\hat{y}\hat{z}\rho_2}{(m_2+\hat{y})^2}+\frac{4\hat{x}\alpha_2\hat{y}\hat{z}^2\rho_2\beta_2}{(m_2+\hat{y})^2}+\frac{2\hat{x}\alpha_2\hat{y}\hat{z}F}{(m_2+\hat{y})^2}-\frac{F^2\alpha_2\hat{z}}{m_2+\hat{y}}-\frac{2\alpha_1\hat{x}\hat{z}\hat{y}\rho_1}{(m_1+\hat{x})^2}+\frac{4\alpha_1\hat{x}\hat{z}^2 y\rho_1\beta_1}{(m_1+\hat{x})^2}-\frac{2\alpha_1\hat{x}\hat{z}\hat{y}\rho_2}{(m_1+\hat{x})^2}+\frac{2\hat{x}\alpha_2\hat{z}\rho_1}{m_2+\hat{y}}-$$
$$\frac{4\hat{x}\alpha_2\hat{z}^2\rho_1\beta_1}{m_2+\hat{y}}+\frac{2\hat{x}\alpha_2\hat{z}\rho_2}{m_2+\hat{y}}-\frac{4\hat{x}\alpha_2\hat{z}^2\rho_2\beta_2}{m_2+\hat{y}}-\frac{2\hat{x}\alpha_2\hat{z}F}{m_2+\hat{y}}-\frac{\alpha_1\hat{z}F^2}{m_1+\hat{x}}+\frac{\alpha_1\hat{x}\hat{z}^2\alpha_2\hat{y}\rho_1}{(m_1+\hat{x})^2(m_2+\hat{y})^2}+\frac{\alpha_1\hat{x}\hat{z}^2\alpha_2\hat{y}\rho_2}{(m_1+\hat{x})^2(m_2+\hat{y})^2}+$$
$$4F\rho_1\hat{z}\beta_1+\frac{\alpha_1\hat{x}\hat{z}\rho_1}{(m_1+\hat{x})^2}-\frac{2\alpha_1\hat{x}\hat{z}^2\rho_1\beta_1}{(m_1+\hat{x})^2}+\frac{2\alpha_1\hat{x}\hat{z}\hat{y}F}{(m_1+\hat{x})^2}+\frac{2\alpha_1\hat{z}^2\rho_1\beta_1}{m_1+\hat{x}}+\frac{2\alpha_1\hat{z}^2\rho_2\beta_2}{m_1+\hat{x}}+\frac{2\alpha_1\hat{z}^3\alpha_2\hat{y}\rho_1\beta_1}{(m_1+\hat{x})(m_2+\hat{y})^2}+$$
$$\frac{2\alpha_1\hat{z}^3\alpha_2\hat{y}\rho_2\beta_2}{(m_1+\hat{x})(m_2+\hat{y})^2}+\frac{\alpha_1\hat{z}^2\alpha_2\hat{y}F}{(m_1+\hat{x})(m_2+\hat{y})^2}+\frac{4\alpha_1\hat{x}\hat{z}^2\hat{y}\rho_2\beta_2}{(m_1+\hat{x})^2}+\frac{2\alpha_1\hat{x}\hat{z}^2F\rho_2\beta_2}{(m_1+\hat{x})^2}-\frac{2F\alpha_2\hat{z}^2\rho_1\beta_1}{m_2+\hat{y}}-\frac{2F\alpha_2\hat{z}^2\rho_2\beta_2}{m_2+\hat{y}}-$$
$$\frac{F\alpha_2\hat{y}\hat{z}\rho_1}{(m_2+\hat{y})^2}-\frac{F\alpha_2\hat{y}\hat{z}\rho_2}{(m_2+\hat{y})^2}-\frac{\alpha_1\hat{x}\hat{z}F\rho_2}{(m_1+\hat{x})^2}-\frac{2\alpha_1\hat{z}^2F\rho_2\beta_2}{m_1+\hat{x}}-2\hat{y}\rho_1-2F^2\hat{x}-2\hat{x}\rho_1-2\hat{y}\rho_2+\frac{\alpha_1\hat{x}\hat{z}F^2}{(m_1+\hat{x})^2}+\frac{F\alpha_2\hat{z}\rho_1}{m_2+\hat{y}}-$$
$$4F\hat{x}\rho_1\hat{z}\beta_1-4F\hat{x}\rho_2\hat{z}\beta_2+4F^2+\frac{2\alpha_1\hat{z}\hat{y}\rho_1}{m_1+\hat{x}}+\frac{2F\alpha_2\hat{y}\hat{z}^2\rho_1\beta_1}{(m_2+\hat{y})^2}+\frac{\alpha_1\hat{z}F}{m_1+\hat{x}}-F^3+\frac{F\alpha_2\hat{z}\rho_2}{m_2+\hat{y}}+\frac{\alpha_2\hat{y}\hat{z}\rho_2}{(m_2+\hat{y})^2}-$$
$$\frac{2\alpha_2\hat{y}\hat{z}^2\rho_2\beta_2}{(m_2+\hat{y})^2}+4F\rho_2\hat{z}\beta_2+\frac{\alpha_1\hat{z}^2\alpha_2\rho_1}{(m_1+\hat{x})(m_2+\hat{y})}-\frac{2\alpha_1\hat{z}^3\alpha_2\rho_1\beta_1}{(m_1+\hat{x})(m_2+\hat{y})}-\frac{\alpha_2\hat{z}\rho_2}{m_2+\hat{y}}-\frac{\alpha_2\hat{z}\rho_1}{m_2+\hat{y}}-8\hat{x}\hat{y}\rho_1\hat{z}\beta_1-$$
$$8\hat{x}\hat{y}\rho_2\hat{z}\beta_2-2F^2\hat{y}+F^2\rho_1+\frac{F\alpha_2\hat{z}}{m_2+\hat{y}}-F+\frac{F^2\alpha_2\hat{y}\hat{z}}{(m_2+\hat{y})^2}-\frac{4\alpha_1\hat{z}^2\hat{y}\rho_1\beta_1}{m_1+\hat{x}}+\frac{2\alpha_1\hat{z}\hat{y}\rho_2}{m_1+\hat{x}}-\frac{4\alpha_1 z^2\hat{y}\rho_2\beta_2}{m_1+\hat{x}}-$$
$$\frac{2\alpha_1\hat{z}\hat{y}F}{m_1+\hat{x}}+\frac{\alpha_1\hat{z}^2\alpha_2\rho_2}{(m_1+\hat{x})(m_2+\hat{y})}-\frac{2\alpha_1\hat{z}^3\alpha_2\rho_2\beta_2}{(m_1+\hat{x})(m_2+\hat{y})}+\frac{\alpha_1\hat{x}\hat{z}\rho_2}{(m_1+\hat{x})^2}-\frac{2\alpha_1\hat{x}\hat{z}^2\rho_2\beta_2}{(m_1+\hat{x})^2}-\frac{2\alpha_1\hat{x}\hat{z}^3\alpha_2\hat{y}\rho_1\beta_1}{(m_1+\hat{x})^2}-$$
$$\frac{2\alpha_1\hat{x}z^3\alpha_2\hat{y}\rho_2\beta_2}{(m_1+\hat{x})^2(m_2+\hat{y})^2}-\frac{\alpha_1\hat{x}\hat{z}^2\alpha_2\hat{y}F}{(m_1+\hat{x})^2(m_2+\hat{y})^2}-\frac{\alpha_1\hat{x}\hat{z}F\rho_1}{(m_1+\hat{x})^2}+\frac{2\alpha_1\hat{x}\hat{z}^2F\rho_1\beta_1}{(m_1+\hat{x})^2}+\frac{\alpha_2 yz\rho_1}{(m_2+y)^2}-\frac{2\alpha_2 yz^2\rho_1\beta_1}{(m_2+y)^2}+4\hat{x}\rho_2\hat{z}\beta_2+$$
$$4\hat{x}\hat{y}\rho_2+2F\hat{x}\rho_2+2F\hat{y}\rho_2+2F\hat{y}\rho_1+4x\hat{y}\rho_1-4\hat{x}\hat{y}F+2F\hat{x}\rho_1-2\hat{x}\rho_2+F^2\rho_2-2F\rho_2+$$
$$2F\hat{y}-2F\rho_1+2F\hat{x}$$

Table 1. Eigenvalues for $\hat{E}$

		m_1 & m_2				
		0,1	**0,2**	**0,5**	**0,9**	**1**
ρ_1	**1**	0,0646616542-0,5976668561i	0,0724812029-0,363397974i	-0,3628643275	-0,5634626514	-0,5910937402
		0,113875598-1,050846561i	0,1915789474-0,854796832i	0,1559808612-0,2390118059i	-0,33276845	-0,3869435658
		-1,9	-1,9	-1,9	-1,9	-1,9
	0,9	0,0638655463-0,5916916521i	0,0705546218-0,3565943472i	-0,3670878594	-0,5656379023	-0,593058764
		0,1133689839-1,044856583i	0,1903529412-0,848394035i	0,1510160427-0,2323946064i	-0,3368299492	-0,3904539415
		-1,7	-1,7	-1,7	-1,7	-1,7
	0,5	0,0571428572-0,5421047417i	0,0542857143-0,2988959276i	-0,3999999996	-0,5829304048	-0,6086739293

		0,1090909091-0,9958591955i	0,18-0,7959899497i	0,1090909092-0,1781447085i	-0,3688199736	-0,4182296668
		-0,9	-0,9	-0,9	-0,9	-0,9
	0,2	0,0285714286-0,3382628692i	0,1148009109	-0,5018614767	-0,6399359837	-0,6603940841
		0,0909090909-0,8132850097i	0,136-0,6000311101i	0,01503152562	-0,4710825966	-0,5083258279
		-0,3	-0,3	-0,3	-0,3	-0,3
	0,1	0,2930433584	0,06322933096	-0,6501596573	-0,7308147263	-0,745079763
		0,0363636363-0,3942655992i	0,0040000001-0,0707546622i	-0,4786996487	-0,6265446806	-0,6482135018
		-0,1	-0,1	-0,1	-0,1	-0,1

The results of analysis show that the equilibrium point$\hat{E}$ is stable. Giving a delay $\tau > 0$ will cause a change in the stability of equilibrium point$\hat{E}$.

3.2 STABILITY OF $\hat{E}$ WITH TIME DELAY

In analyzing the stability of $\hat{E}$ with time delay. It is necessary to linearize equation (3) around the equilibrium point $\hat{E}$, then obtained a linearization model

(a) $\frac{dx(T)}{dT} = b_1 x(T) + b_2 z(T - \tau)$

(b) $\frac{dy(T)}{dT} = b_3 y(T) + b_4 z(T - \tau)$

(c) $\frac{dz(T)}{dT} = b_5 z(T - \tau)$

(5)

where

$b_1 = 1 - 2x - \frac{\alpha_1 z}{m_1 + x} + \frac{\alpha_1 x z}{(m_1 + x)^2} - F, b_2 = \frac{-\alpha_1 x}{m_1 + x}, b_3 = 1 - 2y - \frac{\alpha_2 z}{m_2 + y} + \frac{\alpha_2 y z}{(m_2 + y)^2} - F, b_4 = \frac{-\alpha_2 y}{m_2 + y}, b_5 = \rho_1(1 - \beta_1 z) - \rho_1 z \beta_1 + \rho_2(1 - \beta_2 z) - \rho_2 z \beta_2 - F$

Suppose the solution of the system (5) is

$$x(T) = le^{\lambda\tau}, y(T) = me^{\lambda\tau}, z(T) = ne^{\lambda\tau} \qquad (6)$$

Substitute equation (6) to equation (5) then divided by $e^{\lambda t}$ to obtain

$$\begin{cases} l\lambda = b_1 l + b_2 n e^{-\lambda\tau} \\ m\lambda = b_3 m + b_4 n e^{-\lambda\tau} \\ n\lambda = b_5 n e^{-\lambda\tau} \end{cases} \qquad (7)$$

The system (5) can be written in the following form.

$$\begin{bmatrix} l\lambda \\ m\lambda \\ n\lambda \end{bmatrix} = \begin{bmatrix} b_1 & 0 & b_2 e^{-\lambda\tau} \\ 0 & b_3 & b_4 e^{-\lambda\tau} \\ 0 & 0 & b_5 e^{-\lambda\tau} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} \qquad (8)$$

So that we obtain the following equation characteristic.

$$\begin{vmatrix} b_1-\lambda & 0 & b_2e^{-\lambda\tau} \\ 0 & b_3-\lambda & b_4e^{-\lambda\tau} \\ 0 & 0 & b_5e^{-\lambda\tau}-\lambda \end{vmatrix} = 0$$
$\Leftrightarrow \lambda^3 + \left(-b_1 - b_3 - b_5e^{-\lambda\tau}\right)\lambda^2 + \left(b_1b_3 + b_1b_5e^{-\lambda\tau} + b_3b_5e^{-\lambda\tau}\right)\lambda - b_1b_3b_5e^{-\lambda\tau} = 0$
$\Leftrightarrow \lambda^3 + (-b_1 - b_3)\lambda^2 + b_1b_3\lambda - b_5e^{-\lambda\tau}\lambda^2 + (b_1b_5 + b_3b_5)e^{-\lambda\tau}\lambda - b_1b_3b_5e^{-\lambda\tau} = 0$ (9)
The eigenvalues of equation (9) are real negative and complex number with negative real parts. So with time delay, $\widehat{\mathrm{E}}$ stable if only if $-b_1 - b_3 - b_5 > 0, b_1b_3 + b_1b_5 + b_3b_5 > 0$ dan $-b_1b_3b_5 > 0$. So that the eigenvalues of the Jacobian matrix (9) assumed by $\lambda = u + i\omega$ with $u = 0$ and $\omega > 0(\lambda = +i\omega)$. To see the stability changes in the equation model with the time delay, the eigenvalue substituted into equation (9) so that the roots of the characteristic equation obtained.
$\lambda^3 + (-b_1 - b_3)\lambda^2 + b_1b_3\lambda - b_5e^{-\lambda\tau}\lambda^2 + (b_1b_5 + b_3b_5)e^{-\lambda\tau}\lambda - b_1b_3b_5e^{-\lambda\tau} = 0$
$\Leftrightarrow (i\omega)^3 + (-b_1 - b_3)(i\omega)^2 + b_1b_3(i\omega) - b_5(cos\omega\tau - isin\omega\tau)(i\omega)^2 + (b_1b_5 +$
$b_3b_5)(cos\omega\tau - isin\omega\tau)(i\omega) - b_1b_3b_5(cos\omega\tau - isin\omega\tau) = 0$
$\Leftrightarrow (b_1b_3b_5cos\omega\tau - b_1b_5\omega sin\omega\tau + b_1\omega^3b_3b_5sin\omega\tau - b_3\omega^2 - \omega^2b_5cos\omega\tau + \omega^3) +$
$i(-b_1b_3b_5sin\omega\tau - b_1b_3\omega - b_1b_5\omega cos\omega\tau + b_1\omega^3b_3b_5cos\omega\tau + \omega^2b_5sin\omega\tau) = 0$ (10)
Equation (10) is zero if imaginary and real part are equal to zero so that it obtained
$(b_1b_3b_5 - \omega^2b_5)cos\omega\tau + (b_1\omega^3b_3b_5 - b_1b_5\omega)sin\omega\tau = b_3\omega^2 - \omega^3$ (11)
and $(\omega^2b_5 - b_1b_3b_5)sin\omega\tau + (b_1\omega^3b_3b_5 - b_1b_5\omega)cos\omega\tau = b_1b_3\omega$

(12)

Then eliminating the equations (11) and (12) against τ by squaring each segment of the equation, then obtained
${b_1}^2{b_3}^2{b_5}^2cos^2\omega\tau - 2b_1b_3{b_5}^2\omega^2cos^2\omega\tau + 2{b_1}^2{b_3}^2{b_5}^2\omega^3cos\omega\tau sin\omega\tau -$
$2{b_1}^2b_3{b_5}^2cos\omega\tau sin\omega\tau + \omega^4{b_5}^2cos^2\omega\tau - 2{b_5}^2b_1b_3\omega^5cos\omega\tau sin\omega\tau +$
$2\omega^3{b_5}^2b_1cos\omega\tau sin\omega\tau + {b_1}^2\omega^6{b_3}^2{b_5}^2sin^2\omega\tau - 2{b_1}^2\omega^4b_3{b_5}^2sin^2\omega\tau + {b_1}^2{b_5}^2\omega^2sin^2\omega\tau =$
${b_3}^2\omega^4 - 2b_3\omega^5 + \omega^6$ (13)
$sin^2\omega^4{b_5}^2 - 2sin^2\omega^2{b_5}^2b_1b_3 + 2cos\omega^5{b_5}^2b_1b_3sin\omega\tau - 2\omega^3{b_5}^2b_1cos\omega\tau sin\omega\tau +$
${b_1}^2{b_3}^2{b_5}^2sin^2\omega\tau - 2\omega^3{b_1}^2{b_3}^2{b_5}^2cos\omega\tau sin\omega\tau + 2\omega{b_1}^2b_3{b_5}^2sin\omega\tau cos\omega\tau +$
${b_1}^2{b_3}^2{b_5}^2\omega^6cos^2\omega\tau - 2{b_1}^2b_3{b_5}^2\omega^4cos^2\omega\tau + {b_1}^2{b_5}^2\omega^2cos^2\omega\tau = {b_1}^2{b_3}^2\omega^2$ (14)
The equations (13) and (14) summed and grouped according to rank, then obtained polynomial with the degree of six as follow.
$\left(1 - {b_1}^2{b_3}^2{b_5}^2\right)\omega^6 - 2b_3\omega^5 + \left({b_3}^2 + 2{b_1}^2b_3{b_5}^2 - {b_5}^2\right)\omega^4 + \left(2b_1b_3{b_5}^2 - {b_1}^2{b_5}^2 +\right.$
$\left.{b_1}^2{b_3}^2\right)\omega^2 - {b_1}^2{b_3}^2{b_5}^2 = 0$ (15)
Furthermore, the value of critical delay time (τ_k) with the following stages.
The first step substitutes ω_k into equations (11) and (12) then eliminates the cosine function of equations (11) and (12)
$\left({\omega_K}^6{b_1}^2{b_3}^2{b_5}^2 - 2{b_1}^2b_3{b_5}^2{\omega_K}^4 + {b_1}^2{b_5}^2{\omega_K}^2 - 2{\omega_K}^2{b_5}^2b_1b_3 + {\omega_K}^4{b_5}^2 +\right.$
$\left.{b_1}^2{b_3}^2{b_5}^2\right)sin\omega_K\tau = {b_3}^2{\omega_K}^5b_1b_5 - b_1{\omega_K}^3b_3b_5 - {\omega_K}^6b_1b_3b_5 + {\omega_K}^4b_1b_5 - {b_1}^2{b_3}^2\omega_Kb_5 +$
$b_1{\omega_K}^3b_3b_5$ (16)
Next eliminating the sine function of equations (11) dan (12)
$\left(2{\omega_K}^2b_1b_3{b_5}^2 - {b_1}^2{b_3}^2{b_5}^2 - {b_5}^2{\omega_K}^4 - {b_1}^2{b_3}^2{b_5}^2{\omega_K}^6 + 2{\omega_K}^4{b_1}^2b_3{b_5}^2 -\right.$
$\left.{b_1}^2{b_5}^2{\omega_K}^2\right)cos\omega_K\tau = b_3{\omega_K}^4b_5 - {b_3}^2{\omega_K}^2b_1b_5 - {\omega_K}^5b_5 + {\omega_K}^3b_1b_3b_5 - {b_1}^2{b_3}^2{\omega_K}^4b_5 +$
${b_1}^2b_3{\omega_K}^2b_5$ (17)
From (15) and (16) obtained

$$\tau_k = \frac{1}{\omega_k} tan^{-1}\left(-\frac{{b_3}^2{\omega_K}^5 b_1 b_5 - b_1 {\omega_K}^3 b_3 b_5 - {\omega_K}^6 b_1 b_3 b_5 + {\omega_K}^4 b_1 b_5 - {b_1}^2 {b_3}^2 \omega_K b_5 + b_1 {\omega_K}^3 b_3 b_5}{b_3 {\omega_K}^4 b_5 - {b_3}^2 {\omega_K}^2 b_1 b_5 - {\omega_K}^5 b_5 + {\omega_K}^3 b_1 b_3 b_5 - {b_1}^2 {b_3}^2 {\omega_K}^4 b_5 + {b_1}^2 b_3 {\omega_K}^2 b_5}\right). \quad (18)$$

Further differentiating equation (9) against τ, then obtained

$$\lambda^3 + J\lambda^2 + K\lambda + Le^{-\lambda\tau}\lambda^2 + Me^{-\lambda\tau}\lambda + Ne^{-\lambda\tau} = 0 \quad (19)$$

with $J = -b_1 - b_3, K = b_1 b_3, L = -b_5, M = b_1 b_5 + b_3 b_5, N = -b_1 b_3 b_5$

$$\Leftrightarrow \frac{d(\lambda^3)}{d\lambda}\frac{d\lambda}{d\tau} + J\frac{d(\lambda^2)}{d\lambda}\frac{d\lambda}{d\tau} + K\frac{d\lambda}{d\lambda}\frac{d\lambda}{d\tau} + L\left\{\lambda^2\left[\frac{d(e^{-\lambda\tau})}{d(-\lambda\tau)}\left(-\lambda\frac{d\tau}{d\lambda}\frac{d\lambda}{d\tau} + \tau\frac{d(-\lambda)}{d\lambda}\frac{d\lambda}{d\tau}\right)\right] + e^{-\lambda\tau}\frac{d(\lambda^2)}{d\lambda}\frac{d\lambda}{d\tau}\right\}$$
$$+M\left\{\lambda\left[\frac{d(e^{-\lambda\tau})}{d(-\lambda\tau)}\left(-\lambda\frac{d\tau}{d\lambda}\frac{d\lambda}{d\tau} + \tau\frac{d(-\lambda)}{d\lambda}\frac{d\lambda}{d\tau}\right)\right] + e^{-\lambda\tau}\frac{d\lambda}{d\lambda}\frac{d\lambda}{d\tau}\right\} + N\left[\frac{d(e^{-\lambda\tau})}{d(-\lambda\tau)}\left(-\lambda\frac{d\tau}{d\lambda}\frac{d\lambda}{d\tau} + \tau\frac{d(-\lambda)}{d\lambda}\frac{d\lambda}{d\tau}\right)\right] = 0$$

$$\Leftrightarrow \left(3\lambda^2 + 2J\lambda + K - (L\lambda^2 + M\lambda + N)\tau e^{-\lambda\tau} + (2L\lambda + M)e^{-\lambda\tau}\right)\frac{d\lambda}{d\tau} = (L\lambda^2 + M\lambda + N)\lambda e^{-\lambda\tau}$$

$$\Leftrightarrow \frac{d\lambda}{d\tau} = \frac{(L\lambda^2 + M\lambda + N)\lambda e^{-\lambda\tau}}{3\lambda^2 + 2J\lambda + K - (L\lambda^2 + M\lambda + N)\tau e^{-\lambda\tau} + (2L\lambda + M)e^{-\lambda\tau}}.$$

From equation (19), we have $e^{-\lambda\tau} = \frac{-\lambda^3 - J\lambda^2 - K\lambda}{L\lambda^2 + M\lambda + N}$. Then we get

$$\frac{d\lambda}{d\tau} = \frac{-\lambda^3 - J\lambda^2 - K\lambda}{3\lambda^2 + 2J\lambda + K - (L\lambda^2 + M\lambda + N)\tau e^{-\lambda\tau} + (2L\lambda + M)e^{-\lambda\tau}}.$$

$$Re\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau_b}$$
$$= \frac{\lambda(-\lambda^3 - J\lambda^2 - K\lambda)}{3\lambda^2 + 2J\lambda + K - \tau(-\lambda^3 - J\lambda^2 - K\lambda) + (2L\lambda + M)e^{-\lambda\tau}}$$
$$= \frac{i\omega_b(-(i\omega_b)^3 - J(i\omega_b)^2 - Ki\omega_b)}{3(i\omega_b)^2 + 2Ji\omega_b + K - \tau_b(-(i\omega_b)^3 - J(i\omega_b)^2 - Ki\omega_b) + (2Li\omega_b + M)(cos\,\omega_b\tau_b - i\,sin\,\omega_b\tau_b)}$$
$$= \frac{-{\omega_b}^4 + iA{\omega_b}^3 + B{\omega_b}^2}{-3{\omega_b}^2 + 2Ji\omega_b + K - \tau_b i{\omega_b}^3 - \tau_b J{\omega_b}^2 + \tau_b Ki\omega_b + 2Li\omega_b\,cos\,\omega_b\tau_b + 2L\omega_b\,sin\,\omega_b\tau_b + M\,cos\,\omega_b\tau_b - Mi\,sin\,\omega_b\tau_b}$$

$$= \frac{-{\omega_b}^4 + K{\omega_b}^2 + J{\omega_b}^3 i}{{P_1}^2 + {Q_1}^2}.(P_1 - Q_1 i)$$

with

$$P_1 = -3{\omega_b}^2 + K - \tau_b J{\omega_b}^2 + 2L\omega_b\,sin\,\omega_b\tau_b + M\,cos\,\omega_b\tau_b$$
$$Q_1 = 2J\omega_b - \tau_b{\omega_b}^3 + \tau_b K\omega_b + 2L\omega_b\,cos\,\omega_b\tau_b - M\,sin\,\omega_b\tau_b$$
$$= \frac{3{\omega_b}^6 - 4K{\omega_b}^4 + 2J^2{\omega_b}^4 + K^2{\omega_b}^2}{{P_1}^2 + {Q_1}^2}$$
$$= \frac{{\omega_b}^2(3{\omega_b}^4 + K^2 + (-4K + 2J^2){\omega_b}^2)}{{P_1}^2 + {Q_1}^2}$$

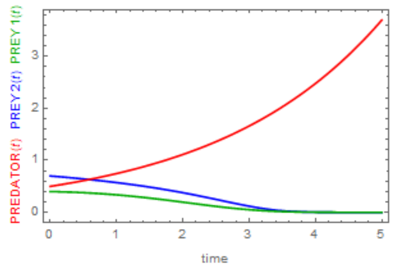
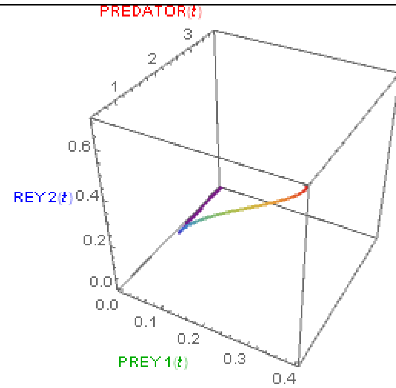
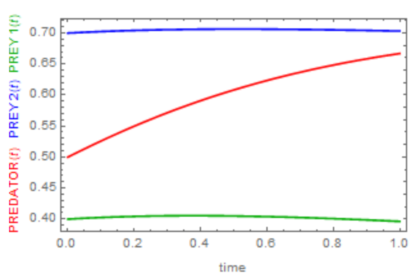
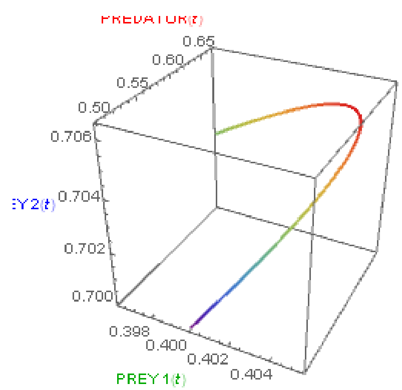
If ω_b is the smallest positive root of (15) (except if it is a twin root then choose the next smallest root), then

$$\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_b} = \frac{{\omega_b}^2(3{\omega_b}^4 + (-4K + 2J^2){\omega_b}^2)}{{P_1}^2 + {Q_1}^2} > 0$$

which shown that the transverse condition is satisfied, so Hopf bifurcation occur at $\tau = \tau_b$.

4.3 SIMULATION WITHOUT TIME DELAY

In this section, we give the model simulations at $E_0, E_1, E_2, E_3, E_4, E_5, E_6, \hat{E}, E_8, E_9$ dan E_{10}. To find the solution, we select the parameter values which are given as follows

E_0 E_1 E_2 E_3 E_4 E_5 E_6 $\hat{E}$	$\alpha_1 = 0{,}7$ $\alpha_2 = 1{,}1$ $m_1 = 0{,}3$ $m_2 = 0{,}6$ $\rho_1 = 0{,}5$ $\rho_2 = 0{,}5$ $\beta_1 = 1{,}8$ $\beta_2 = 1{,}8$ $F = 0{,}1$		
E_8 E_9 E_{10}	$\alpha_1 = 0{,}7$ $\alpha_2 = 1{,}1$ $m_1 = 0{,}3$ $m_2 = 0{,}6$ $\rho_1 = 0{,}5$ $\rho_2 = 0{,}5$ $\beta_1 = 1{,}8$ $\beta_2 = 1{,}8$		

$F =$
0,1

Figure 1. Solution fields and phase portrait

4.4 SIMULATION AT $\hat{E}$ WITH TIME DELAY

The numerical simulations of the predator-prey model with time-delay is done to show the effect of time-delay on the stability of $\hat{E}$.

Table 2. Parameter values at $\hat{E}$ with time delay

Parameter	Value
α_1	0,7
α_2	1,1
m_1	1,5
m_2	0,9
ρ_1	1,9
ρ_2	0,5
β_1	1,8
β_2	1,8
F	0,1

From Table 2 obtained

$$A = \sqrt{\frac{\left(F-(m_1+1)\right)^2(\rho_1\beta_1+\rho_2\beta_2)-4\alpha_1(\rho_1+\rho_2-F)}{(\rho_1\beta_1+\rho_2\beta_2)^2}} = 0{,}9941098824$$

$$B = \sqrt{\frac{\left(F-(m_2+1)\right)^2(\rho_1\beta_1+\rho_2\beta_2)-4\alpha_2(\rho_1+\rho_2-F)}{(\rho_1\beta_1+\rho_2\beta_2)^2}} = 0{,}4557775732$$

$$x = -\frac{1}{2}(F - 1 + m_1 - A) = 0{,}7331092947.$$

$$y = y = -\frac{1}{2}(F - 1 + m_1 - B) = 0{,}4736579482.$$

$$z = \frac{\rho_1+\rho_2-F}{\rho_1\beta_1+\rho_2\beta_2} = 0{,}5324074074.$$

So we get $\hat{E}(0{,}7331092947; 0{,}4736579482; 0{,}5324074074)$.

Then from the parameter values presented in Table 2, it is obtained

$$b_1 = 1 - 2x - \frac{\alpha_1 z}{m_1+x} + \frac{\alpha_1 xz}{(m_1+x)^2} - F = -0{,}6783206068$$

$$b_2 = \frac{-\alpha_1 x}{m_1+x} = -0{,}2298035781$$

$$b_3 = 1 - 2y - \frac{\alpha_2 z}{m_2+y} + \frac{\alpha_2 yz}{(m_2+y)^2} - F = -0{,}3266487880$$

$$b_4 = \frac{-\alpha_2 y}{m_2+y} = -0{,}3792965663$$

$$b_5 = \rho_1(1-\beta_1 z) - \rho_1 z\beta_1 + \rho_2(1-\beta_2 z) - \rho_2 z\beta_2 - F = -2{,}3$$

$$\left(1 - {b_1}^2{b_3}^2{b_5}^2\right)\omega^6 - 2b_3\omega^5 + \left({b_3}^2 + 2{b_1}^2 b_3 {b_5}^2 - {b_5}^2\right)\omega^4 + \left(2b_1 b_3 {b_5}^2 - {b_1}^2{b_5}^2 + {b_1}^2{b_3}^2\right)\omega^2 - {b_1}^2{b_3}^2{b_5}^2 = 0$$

$$\Leftrightarrow 0{,}7402905241\omega^6 + 0{,}653297576\omega^5 - 6{,}773445615\omega^4 - 0{,}040696122\omega^2 - 0{,}2597094759 = 0. \quad (30)$$

Because $\omega > 0$, then chosen$\omega_1 = 2{,}618150518$. Then we can find the value of τ_k by substituting the values of b_1, b_2, b_3, b_4, b_5 and ω_k in the following equation.

$$\Leftrightarrow \tau_k = \frac{1}{\omega_k} tan^{-1}\left(-\frac{{b_3}^2{\omega_K}^5 b_1 b_5 - b_1{\omega_K}^3 b_3 b_5 - {\omega_K}^6 b_1 b_3 b_5 + {\omega_K}^4 b_1 b_5 - {b_1}^2{b_3}^2\omega_K b_5 + b_1{\omega_K}^3 b_3 b_5}{b_3{\omega_K}^4 b_5 - {b_3}^2{\omega_K}^2 b_1 b_5 - {\omega_K}^5 b_5 + {\omega_K}^3 b_1 b_3 b_5 - {b_1}^2{b_3}^2{\omega_K}^4 b_5 + {b_1}^2 b_3{\omega_K}^2 b_5}\right).$$

$$\Leftrightarrow \tau_k = \frac{1}{2{,}618150518} tan^{-1}\left(-\frac{258{,}2193419}{315{,}6334948} + \pi\right).$$

$$\Leftrightarrow \tau_k = 0{,}9380335222.$$

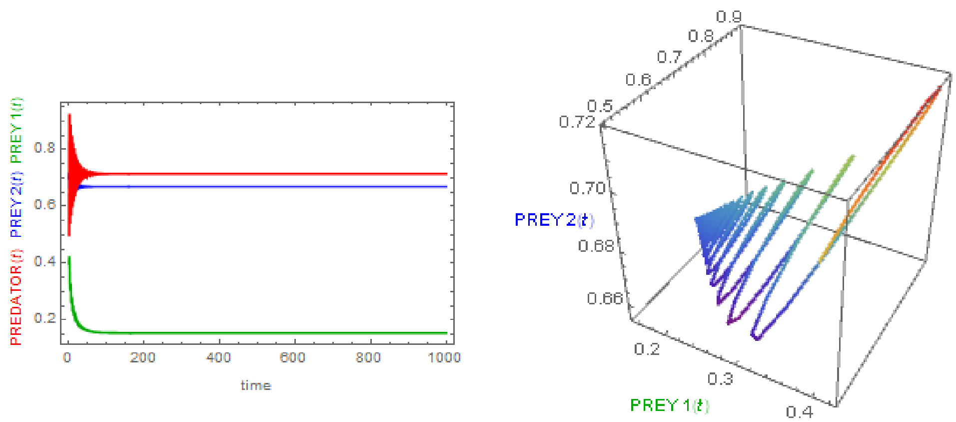
In table 3 it can be seen that the delay timeout value at a distance of $k = 0, 1, 2, 3, \ldots, n$.

Table 3. Time Delay at $k = 0, 1, 2, 3, \ldots, n$

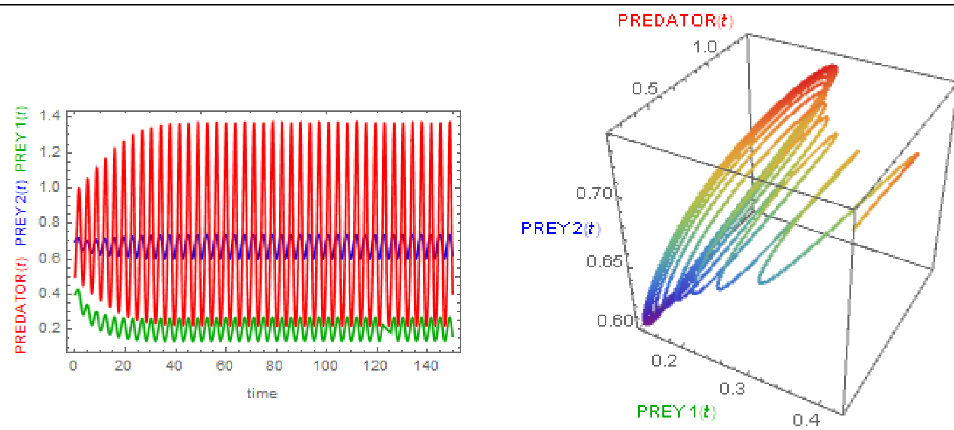
$\boldsymbol{k}$	$\boldsymbol{\tau}$
0	$-0{,}2618946836$
1	$0{,}9380335222$
2	$2{,}137961728$
⋮	⋮
n	τ_n

In this article, we only discuss the value of the delay time at, before and after the delay time at $k = 1$. In this simulation, three cases will be provided to show the existence of the Hopf bifurcation.

$\tau = 0{,}75 < \tau_k$



$\tau = 0{,}93 = \tau_k$



$\tau = 0{,}95 > \tau_k$

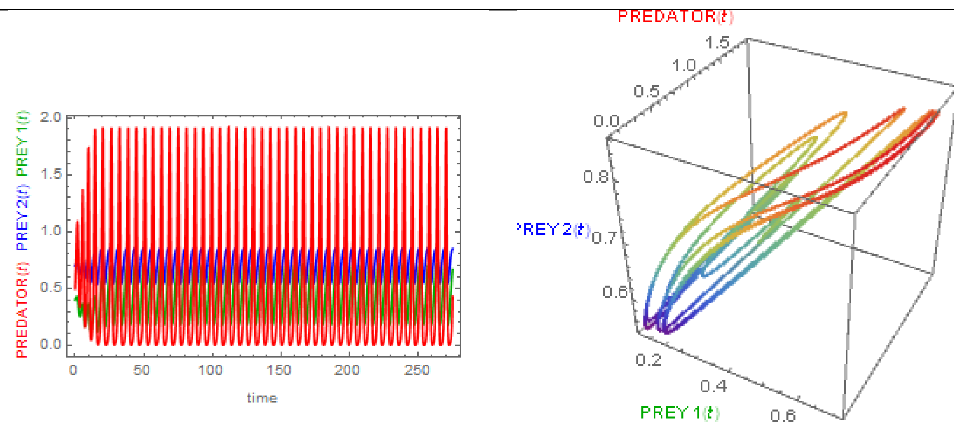


Figure 2. Solution fields and phase portrait at $\hat{E}$

5. CONCLUSION

From the above discussion, can be concluded that be based on the non-dimensional model, we obtain the following mathematical model in a modified Leslie-Gower system with Holling type II functional response with harvesting and time delay.

From that model, eleven equilibrium points are obtained, i.e. $E_0, E_1, E_2, E_3, E_4, E_5, E_6, \hat{E}, E_8, E_9$ and E_{10} with the assumptions there are the equilibrium points $E_4, E_5, E_6, \hat{E}, E_8, E_9$ and E_{10} if $\rho_1 + \rho_2 > F, F - 1 + m_1 \pm A < 0$ and $F - 1 + m_1 \pm B < 0$.

From the stability analysis, we obtain eight equilibrium points which can be stable in the certain condition, i.e. the equilibrium points $E_4, E_5, E_6, \hat{E}$ stable if $(A + \alpha_1)(-m_1 + F - 1 + A)^2(\rho_1\beta_1 + \rho_2\beta_2) + 4\alpha_1 m_1(-\rho_1 - \rho_2 + F) < 0, \alpha_2(\rho_1 + \rho_2 - F) + m_2(F - 1)(\rho_1\beta_1 + \rho_2\beta_2) > 0, \alpha_1(\rho_1 + \rho_2 - F) + m_1(F - 1)(\rho_1\beta_1 + \rho_2\beta_2) > 0$ and $(B + m_1)(-2m_2 + F - 1 + m_1 + B)^2(\rho_1\beta_1 + \rho_2\beta_2) + 4\alpha_2 m_2(F - \rho_2 - \rho_1) < 0$. E_8, E_9 and E_{10} stable if $(A - m_1)(\rho_1\beta_1 + \rho_2\beta_2)(m_1 - F + 1 + A)^2 > 4\alpha_1 m_1(-\rho_1 - \rho_2 + F), (B - m_1)(\rho_1\beta_1 + \rho_2\beta_2)(2m_2 - F + 1 - m_1 + B)^2 > 4\alpha_2 m_2(-\rho_1 - \rho_2 + F), \alpha_1(\rho_1 + \rho_2 - F) + Fm_1(\rho_1\beta_1 + \rho_2\beta_2) > m_1(\rho_1\beta_1 + \rho_2\beta_2)$ and $\alpha_2(\rho_1 + \rho_2 - F) + Fm_2(\rho_1\beta_1 + \rho_2\beta_2) > m_2(\rho_1\beta_1 + \rho_2\beta_2)$.

Models with harvest often link population with economic problems. The effect of harvest level on the results shows that catch quotas can cause oscillations, chaos and increase the risk of exploitation.

In analyzing the existence of Hopf bifurcation, this model is divided into three cases where each case experiences an increase in the value of time delay parameter in predator population (τ). In the case of $\tau = \tau_k$ there is a change in the stability of interior point from a stable spiral to an unstable spiral and a Limit Cycle appears. This phenomenon is a property of the Hopf bifurcation. By selecting the appropriate bifurcation parameters, we investigated the local stability and the Hopf bifurcation. Observations on the model simulation are carried out by varying the value of delay time. When the Hopf bifurcation occurs, the graph on the solution plane shows a constant oscillatory movement. If $\tau < \tau_k$, the controlled system solution is in a state of balance. Then when $\tau > \tau_k$ the system solution continues to fluctuate causing an unstable system condition.

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