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Continuous weaving g-frames in Hilbert C^*-modules

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Abstract.

The present paper aims to study c-g-woven for Hilbert C^*-modules, first, we give some definitions and fundamental properties which will be useful to introduce this notion. And also some of his properties are given. Finally, we discuss the perturbation for c-g-woven.

Keywords: Continuous woven g-frames, g-frames, Hilbert C^*-modules.

1.Introduction and Preliminaries

Frames generalise orthonormal bases and were introduced by Duffin and Schaefer [7] in 1952 to analyse some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [10] for signal processing. In 2000, Frank-larson [8] introduced the concept of frames in Hilbet C^*-modules as a generalization of frames in Hilbert spaces. The basic idea was to consider modules over C^*-algebras of linear spaces and to allow the inner product to take values in the C^*-algebras [14]. A. Khosravi and B. Khosravi [12] introduced the fusion frames and $g-$frame theory in Hilbert C^*-modules. Afterwards, A. Alijani and M. Dehghan consider frames with C^*-valued bounds [2] in Hilbert C^*-modules. N. Bounader and S.

Kabbaj [5] and A. Alijani [1] introduced the $*$-g-frames which are generalizations of g-frames in Hilbert C^*-modules. In 2016, Z. Xiang and Y. Li [26] give a generalization of $g-$frames for operators in Hilbert C^*-modules. Recently, Fakhr-dine Nhari et al. [15] introduced the concepts of g-fusion frame and K-g-fusion frame in Hilbert C^*-modules. Bemrose et al. [4] introduced a new concept of weaving frames in separable Hilbert spaces. This notion has potential applications in distributed signal processing and wireless sensor networks. Weaving Frames in Hilbert C^*-Modules introduced by X. Zhao and P. Li [29]. For more on frame in Hilbert C^*-modules see [11, 18, 19, 20, 21, 22] and references therein.

We organize the rest of the paper as follows. We continue with this section to collect some definitions, and basic lemmas which will be used in next. In section 2 we introduce the notion of continuous g-woven in Hilbert C^*-module and discuss some of their properties. In section 3 we discuss perturbation of c-g-woven.

Throughout this paper, (Ω, μ) is a measure space with positive measure μ, $\mathcal{H}$ and $\{H_\omega\}_{\omega\in\Omega}$ are in Hilbert C^*-module and a family of Hilbert C^*-module, respectively, and $U \in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$ is the set of all adjointable operators from $\mathcal{H}$ into $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$, then $End^*_{\mathcal{A}}(\mathcal{H},\mathcal{H})$ will be denoted by $\mathcal{H}$. For each $m > 1$ where $m \in \mathbb{N}$, we define $[m] := \{1, 2, \dots, m\}$ and $[m]^c = \{m + 1, m + 2, \dots\}$.

Definition 0.1. [16] Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-hilbert $\mathcal{A}$-Module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued in product $\langle .,. \rangle_A$:$\mathcal{H}\times\mathcal{H} \to \mathcal{A}$ such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x\rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x\rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.
(ii) $\langle ax + y, z\rangle_{\mathcal{A}} = a\langle x, z\rangle_{\mathcal{A}} + \langle y, z\rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.
(iii) $\langle x, y\rangle_{\mathcal{A}} = \langle y, x\rangle^*_{\mathcal{A}}$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$ we define $\|x\|= \|\langle x, x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $\|.\|$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$.
For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = (x^*x)^{\frac{1}{2}}$ for $x \in \mathcal{H}$.
Let $\mathcal{H}$ and $\mathcal{K}$ be tow Hilbert $\mathcal{A}$ modules, A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ for all $y \in \mathcal{K}$ and $x \in \mathcal{H}$.

Definition 0.2. [27] Let $T \in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$, then a bounded adjointable operator $T^\dagger \in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$ is called the Moore–Penrose inverse of T if $TT^\dagger T = T, T^\dagger TT^\dagger = T^\dagger, \left(TT^\dagger\right)^* = TT^\dagger$ and $\left(T^\dagger T\right)^* = T^\dagger T$.

The notation $T^\dagger$ is reserved to denote the Moore-Penrose inverse of T. These properties imply that $T^\dagger$ is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections. Moreover, Ran $\left(T^\dagger\right) =$ Ran $\left(T^\dagger T\right)$, Ran$(T) =$ Ran $\left(TT^\dagger\right)$, Ker$(T) =$ Ker $\left(T^\dagger T\right)$ and Ker $\left(T^\dagger\right) =$ Ker $\left(TT^\dagger\right)$ which lead us to $\mathcal{K} = \text{Ker}\left(T^\dagger T\right) \oplus \text{Ran}\left(T^\dagger T\right) = \text{Ker}(T) \oplus \text{Ran}\left(T^\dagger\right)$ and $\mathcal{H} = \text{Ker}\left(T^\dagger\right) \oplus \text{Ran}(T)$.

In [27] Xu and Sheng showed that a bounded adjointable operator between two Hilbert C^* modules admits a bounded Moore-Penrose inverse if and only if the operator has closed range, we refer the readers to [9, 23, 24] for more detailed information.

Lemma 0.3. [16] *Let $\mathcal{H}$ and $\mathcal{K}$ two Hilbert $\mathcal{A}$-module and $T \in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$. Then, the following assertions are equivalent:*

(i) *The operator T is bounded and $\mathcal{A}$-linear,*

(ii) *There exist $k > 0$ such that $\langle Tx, Tx\rangle_{\mathcal{A}} \leq k\langle x, x\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$.*

Lemma 0.4. [3]*. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$. Then the following statements are equivalent:*

(i) *T is surjective.*

(ii) *T^* is bounded below with respect to norm, i.e., there is $\tau > 0$ such that $\|T^*x\| \geq \tau\|x\|$, for all $x \in \mathcal{K}$.*

(iii) *T^* is bounded below with respect to the inner product, i.e., there is $\zeta > 0$ such that $\langle T^*x, T^*x\rangle_{\mathcal{A}} \geq \zeta\langle x, x\rangle_{\mathcal{A}}$, for all $x \in \mathcal{K}$.*

Lemma 0.5. [28] *Let (Ω, μ) be a measure space, X and Y are tow Banach spaces, $\lambda : X \to Y$ be a bounded linear operator and $f : \Omega \to Y$ measurable function, then*

$$\lambda\left(\int_\Omega f d\mu\right) = \int_\Omega (\lambda f) d\mu.$$

The following definitions were introduced by M. Rossafi et al. in the paper entitled "Integral frame in Hilbert C^*-module" (see [17]). Let (Ω, μ) be a measure space, $\mathcal{H}$ and V be two Hilbert C^*-modules over a unital C^*-algebra and $\{\mathcal{H}_w\}_{w\in\Omega}$ is a family of submodules of $End^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_w)$ is the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{H}$ into $\mathcal{H}_w$.

We define, following:

$$l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega}) = \left\{F = \{F_w\}_{w\in\Omega} : F_w \in \mathcal{H}_w, \left\|\int_\Omega |F_w|^2 d\mu(w)\right\| < \infty\right\}.$$

For any $F = \{F_w\}_{w\in\Omega}$ and $G = \{G_w\}_{w\in\Omega}$, the $\mathcal{A}$-valued inner product is defined by $\langle F, G\rangle = \int_\Omega \langle F_w, G_w\rangle_{\mathcal{A}} d\mu(w)$ and the norm is defined by $\|F\| = \|\langle F, F\rangle\|^{\frac{1}{2}}$. In this case the $l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega})$ is an Hilbert C^*-module (see [14]).

Definition 0.6 ([13])**.** A family $\Lambda := \{\Lambda_\omega \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_\omega)\}_{\omega\in\Omega}$ is called a continuous g-frame (or briefly $c-g-$ frame) in Hilbert $\mathcal{A}$ module $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ if

(i) the mapping

$$\Omega \longmapsto \mathcal{A}$$

$$\omega \longmapsto \langle \Lambda_\omega f, \Lambda_\omega f\rangle_{\mathcal{A}}$$

is measurable for any $f \in \mathcal{H}$.

(ii) there exist constants $0 < A \leq B < +\infty$ such that for each $f \in \mathcal{H}$,

$$A\langle f, f\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_\omega f, \Lambda_\omega f\rangle_{\mathcal{A}} d\mu(\omega) \leq B\langle f, f\rangle_{\mathcal{A}}. \tag{1}$$

If A, B can be chosen such that $A = B$, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a tight $c-g-$ frame and if $A = B = 1$, it is called parseval c-g-frame. A family $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called $c-g-$ Bessel family if the right hand inequality (1) holds and the number B is called the Bessel constant.
The continuous $g-$ frame operator S_Λ on $\mathcal{H}$ is defined by:

$$S_\Lambda(f) = \int_\Omega \Lambda^*_w \Lambda_w f d\mu(w), \qquad f \in \mathcal{H}.$$

Theorem 0.7. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous $g-$ Bessel family in $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ with the bound B. Then the mapping T_Λ on $l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega})$ to $\mathcal{H}$ is defined by*

$$\langle T_\Lambda F, g\rangle_{\mathcal{A}} = \int_\Omega \langle \Lambda^*_\omega F(\omega), g\rangle_{\mathcal{A}} d\mu(\omega), \qquad F \in l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega}), g \in \mathcal{H},$$

is linear and bounded with $\|T_\Lambda\| \leq \sqrt{B}$. Furthermore, for each $g \in \mathcal{H}$ and $\omega \in \Omega$,

$$T^*_\Lambda(g)(\omega) = \Lambda_\omega(g).$$

In the continuous $g-$ frame operator $S_\Lambda = T_\Lambda T^*_\Lambda$ is defined by

$$S_\Lambda : \mathcal{H} \longrightarrow \mathcal{H},$$

$$\langle S_\Lambda f, g\rangle_{\mathcal{A}} = \int_\Omega \langle f, \Lambda_\omega \Lambda^*_\omega g\rangle_{\mathcal{A}} d\mu.$$

Therefore,

$$AI \leq S_\Lambda \leq BI$$

and we obtain, if $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a $c-g-$ frame, then S_Λ is positive, self-adjoint and invertible operator.

Proof. For any $F \in l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega})$

$$\begin{aligned}\|T_\Lambda F\| = \sup_{\|g\|=1} \|\langle T_\Lambda F, g\rangle_{\mathcal{A}}\| &= \sup_{\|g\|=1} \left\| \int_\Omega \langle \Lambda_\omega^* F(w), g\rangle_{\mathcal{A}} \, d\mu \right\| \\ &= \sup_{\|g\|=1} \left\| \int_\Omega \langle F(w), \Lambda_\omega g\rangle_{\mathcal{A}} \, d\mu \right\| \\ &\leq \sup_{\|g\|=1} \left\| \int_\Omega \langle F, F\rangle_{\mathcal{A}} d\mu \right\|^{\frac{1}{2}} \left\| \int_\Omega \langle \Lambda_\omega g, \Lambda_\omega g\rangle_{\mathcal{A}} \, d\mu \right\|^{\frac{1}{2}} \\ &\leq \sqrt{B} \, \|F\| \, .\end{aligned}$$

Next, we show that S_Λ is a self-adjoint operator

$$\langle S_\Lambda f, g\rangle_{\mathcal{A}} = \langle T_\Lambda T_\Lambda^* f, g\rangle_{\mathcal{A}} = \langle f, T_\Lambda T_\Lambda^* g\rangle_{\mathcal{A}} = \langle f, S_\Lambda g\rangle_{\mathcal{A}}$$

Or, by using the notation from opertor theory, we get

$$AI \leq S_\Lambda \leq BI,$$

thus S_Λ is a positive operator furthermore

$$0 \leq I - B^{-1} S_\Lambda \leq \frac{B-A}{B} I,$$

and consequently

$$\|I - B^{-1} S_\Lambda\| = \sup_{\|f\|=1} \left\| \left\langle I - B^{-1} S_\Lambda f, f \right\rangle_{\mathcal{A}} \right\| \leq \frac{B-A}{B} \leq 1,$$

which shows that S_Λ is inverstible. □

The following Definition is a generalization of the Definition 3.1 in [25], for Hilbert C^*-modules.

Definition 0.8. A family of c-frames $\{F_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ in Hilbert C^*-module $\mathcal{H}$ with respect to μ is said to be c-woven if there exist universal same positive constants $0 < A \leq B < \infty$ such that for each partition $\{\sigma_i\}_{i\in[m]}$ of Ω, the family $\{F_{\omega_i}\}_{\omega\in\sigma_i, i\in[m]}$ is a $c-$ frame in Hilbert C^*-module $\mathcal{H}$ with bounds A and B. Each family $\{F_{\omega_i}\}_{\omega\in\sigma_i, i\in[m]}$ is called a weaving.

2. Continuous weaving g-frames

In this section, we introduce the notation of continuous g-woven in Hilbert C^*-module and discuss some of their properties.

Definition 0.9. Two c-g-frames $\Lambda := \{\Lambda_{\omega_i} \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_\omega)\}_{\omega\in\Omega, i\in[m]}$ and $\Gamma := \{\Gamma_{\omega_i} \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_\omega)\}_{\omega\in\Omega, i\in[m]}$ are said to be continuous g-woven (or $c-g-$woven) if there exist universal constants $0 < A \le B < \infty$ such that for each partition $\{\sigma_i\}_{i\in[m]}$ of Ω, the family $\{\Lambda_{\omega_i}\}_{\omega\in\sigma_i, i\in[m]} \cup \{\Gamma_{\omega_i}\}_{\omega\in\sigma_i, i\in[m]}$ is a $c-g-$ frame in Hilbert C^*-module $\mathcal{H}$ with bounds A and B, respectively, that is

$$A\langle f, f\rangle_{\mathcal{A}} \le \int_{\sigma} \langle \Lambda_{\omega_i} f, \Lambda_{\omega_i} f\rangle_{\mathcal{A}} d\mu(\omega) + \int_{\sigma^c} \langle \Gamma_{\omega_i} f, \Gamma_{\omega_i} f\rangle_{\mathcal{A}} d\mu(\omega) \le B\langle f, f\rangle_{\mathcal{A}}. \tag{2}$$

In the above definition, A and B are called universal c-g-frames bounds.
It is easy to show that every c-g-woven has an universal upper c-g-frame bound.
Indeed, let $\{\Lambda_\omega\}_{\omega\in\Omega}$ is be a c-g-Bessel family in Hilbert C^*-module $\mathcal{H}$ with bounds B_i for each $i \in [m]$. Then, for any partition $\{\sigma_i\}_{i\in[m]}$ of Ω and $f \in \mathcal{H}$, we have

$$\sum_{i\in[m]} \int_{\sigma_i} \langle \Lambda_{\omega_i} f, \Lambda_{\omega_i} f\rangle_{\mathcal{A}} d\mu \le \sum_{i\in[m]} \int_{\Omega} \langle \Lambda_{\omega_i} f, \Lambda_{\omega_i} f\rangle_{\mathcal{A}} d\mu \le (\sum_{i\in[m]} B_i)\langle f, f\rangle_{\mathcal{A}}.$$

In the next results, we construct a c-g-woven by using a bounded linear operator.

Theorem 0.10. *Let $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ be a c-g- woven in Hilbert C^*-module $\mathcal{H}$ with universal bounds A, B. If $T \in End^*_{\mathcal{A}}(\mathcal{H})$ has closed range, then $\{\Lambda_{\omega_i} T^*\}_{\omega\in\Omega, i\in[m]}$ is a $c-g-$ woven for $\mathcal{R}(T)$ with frame bounds $A\|T^\dagger\|^{-2}$ and $B\|T\|^2$.*

Proof. First, since $T^* f \in \mathcal{H}$ and $\omega \longmapsto \langle \Lambda_{\omega_i} f, \Lambda_{\omega_i} f\rangle_{\mathcal{A}}$, is measurable for any $f \in \mathcal{H}$ and $i \in [m]$,then $\omega \longmapsto \langle \Lambda_{\omega_i} T^* f, \Lambda_{\omega_i} T^* f\rangle_{\mathcal{A}}$, is measurable for any $f \in \mathcal{H}$ and $i \in [m]$. On the other hand, for each $f \in \mathcal{R}(T)$, we have

$$\begin{aligned} A\,\|\langle f, f\rangle_{\mathcal{A}}\| &= A\left\|\langle TT^\dagger f, TT^\dagger f\rangle_{\mathcal{A}}\right\| \\ &= A\|TT^\dagger f\|^2 \\ &= A\|(T^\dagger)^* T^* f\|^2 \\ &\le A\|(T^\dagger)\|^2 \|T^* f\|^2 \\ &\le A\|(T^\dagger)\|^2 \|\langle T^* f, T^* f\rangle_{\mathcal{A}}\| \\ &\le \|(T^\dagger)\|^2 \left\|\sum_{i\in[m]} \int_{\Omega} \langle \Lambda_{\omega_i} T^* f, \Lambda_{\omega_i} T^* f\rangle_{\mathcal{A}} d\mu\right\|. \end{aligned}$$

So,

$$\begin{aligned} A\|(T^{\dagger})\|^{-2}\,\|\langle f,f\rangle_{\mathcal{A}}\| &\leq \left\| \sum_{i\in[m]} \int_{\Omega} \langle \Lambda_{\omega_i} T^* f, \Lambda_{\omega_i} T^* f\rangle_{\mathcal{A}} d\mu \right\| \\ &\leq \sum_{i\in[m]} B_i \,\|\langle T^* f, T^* f\rangle_{\mathcal{A}}\| \\ &\leq B\|T\|^2 \|\langle f,f\rangle_{\mathcal{A}}\|\,. \end{aligned}$$

□

The following result presents a relationship between the norms of the c-g-frame operator of the original c-g-frame and the weaving.

Theorem 0.11. *Let $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ be a c-g-woven in $\mathcal{H}$ with universal bounds A and B. If $S_{\Lambda}^{(i)}$ is the c-g-frame operator of $\{\Lambda_{\omega_i}\}_{\omega\in\Omega}$ for each $i \in [m]$, S_{Λ,σ_i} represents the c-g-frame operator of $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ for each partition $\{\sigma_i\}_{i\in[m]}$ of and $S_{\Lambda,\sigma_i}^{(i)}$ denotes the c-g-frame operator $S_{\Lambda}^{(i)}$ with integral restricted to $\{\sigma_i\}$, then for each $f \in \mathcal{H}$,*

$$\sum_{i\in[m]} \|\langle\, S_{\Lambda,\sigma_i}^{(i)} f, S_{\Lambda,\sigma_i}^{(i)} f\rangle_{\mathcal{A}}\| \leq B\|S_{\Lambda,\sigma_i}\|\|\langle f,f\rangle_{\mathcal{A}}\|.$$

Proof. Suppose that $f \in \mathcal{H}$. We can write

$$\begin{aligned} \sum_{i\in[m]} \|\langle\, S_{\Lambda,\sigma_i}^{(i)} f, S_{\Lambda,\sigma_i}^{(i)} f\rangle_{\mathcal{A}}\| &= \sum_{i\in[m]} \|S_{\Lambda,\sigma_i}^{(i)} f\|^2 \\ &= \sum_{i\in[m]} (\sup_{\|g\|=1} \|\langle S_{\Lambda,\sigma_i}^{(i)} f, g\rangle_{\mathcal{A}}\|)^2 \\ &= \sum_{i\in[m]} (\sup_{\|g\|=1} \|\langle T_{\Lambda,\sigma_i}^{(i)} (T_{\Lambda,\sigma_i}^{(i)})^* f, g\rangle_{\mathcal{A}}\|)^2 \\ &\leq \sum_{i\in[m]} B\|\langle (T_{\Lambda,\sigma_i}^{(i)})^* f, (T_{\Lambda,\sigma_i}^{(i)})^* f\rangle_{\mathcal{A}}\| \\ &= B\sum_{i\in[m]} \|\int_{\sigma_i} \langle \Lambda_{\omega_i} f, \Lambda_{\omega_i} f\rangle_{\mathcal{A}} d\mu(\omega)\| \\ &\leq B\|\langle S_{\Lambda,\sigma_i} f, f\rangle_{\mathcal{A}}\| \\ &\leq B\|S_{\Lambda,\sigma_i}\|\|\langle f,f\rangle_{\mathcal{A}}\|. \end{aligned}$$

□

Theorem 0.12. *Let $\Omega_i \subseteq \Omega$ be measurable subsets for all $i \in [m]$, and let F_i and G_i be c-frame mappings on Ω_i for $\mathcal{H}_\omega$ with the pair frame bounds (A_{F_i}, B_{F_i}) and (A_{G_i}, B_{G_i}) respectively, for each $\omega \in \Omega$. Assume that $\Lambda_{\omega_i}, \Theta_{\omega_i} \in End^*_{\mathcal{A}}(\mathcal{H}, \Omega_i)$, for any $i \in [m]$ such that $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ and $\{\Theta_{\omega\in\Omega, i\in[m]}$ are strongly measurable. Then the following assertions are equivalent.*

(I) $\{\Lambda^*_{\omega_i} F_i\}_{\omega\in\Omega, i\in[m]}$ *and* $\{\Theta^*_{\omega_i} G_i\}_{\omega\in\Omega, i\in[m]}$ *are c− woven in Hilbert C^*-module $\mathcal{H}$.*

(II) $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ *and* $\{\Theta_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ *are $c-g-$ woven in Hilbert C^*-module $\mathcal{H}$.*

Proof. $(I) \Rightarrow (II)$. Suppose that $\sigma \subseteq \Omega$ is measurable subsets and $f \in \mathcal{H}$. Let $\{\Lambda^*_{\omega_i} F_i\}_{\omega\in\Omega, i\in[m]}$ and $\{\Theta^*_{\omega_i} G_i\}_{\omega\in\Omega, i\in[m]}$ are c-woven in $\mathcal{H}$, with universal frame bounds C, D and $A = \inf\{A_{F_i}, A_{G_i}\}$. Then, for each $i \in [m]$, we have

$$\begin{aligned}
&A \int_\sigma \langle \Lambda_{\omega_i} f, \Lambda_{\omega_i} f\rangle_{\mathcal{A}} d\mu(\omega) + A \int_{\sigma^c} \langle \Theta_{\omega_i} f, \Theta_{\omega_i} f\rangle_{\mathcal{A}} d\mu(\omega) \\
&\leq \int_\sigma A_{F_{\omega_i}} \langle \Lambda_{\omega_i} f, \Lambda_{\omega_i} f\rangle_{\mathcal{A}} d\mu(\omega) + \int_{\sigma^c} A_{G_{\omega_i}} \langle \Theta_{\omega_i} f, \Theta_{\omega_i} f\rangle_{\mathcal{A}} d\mu(\omega) \\
&\leq \int_\sigma \int_{\Omega_i} |\langle \Lambda_{\omega_i} f, F_i(x)\rangle_{\mathcal{A}}|^2 d\mu(x) d\mu(\omega) + \int_{\sigma^c} \int_{\Omega_i} |\langle \Theta_{\omega_i} f, G_i(x)\rangle_{\mathcal{A}}|^2 d\mu(x) d\mu(\omega) \\
&= \int_\sigma \int_{\Omega_i} |\langle f, \Lambda^*_{\omega_i} F_i(x)\rangle_{\mathcal{A}}|^2 d\mu(x) d\mu(\omega) + \int_{\sigma^c} \int_{\Omega_i} |\langle f, \Theta^*_{\omega_i} G_i(x)\rangle_{\mathcal{A}}|^2 d\mu(x) d\mu(\omega) \\
&\leq D\langle f, f\rangle_{\mathcal{A}}.
\end{aligned}$$

With the same way, we conclude that

$$\begin{aligned}
&B \int_\sigma |\langle \Lambda_{\omega_i} f, \Lambda_{\omega_i} f\rangle_{\mathcal{A}}|^2 d\mu(\omega) + B \int_{\sigma^c} |\langle \Theta_{\omega_i} f, \Theta_{\omega_i} f, \rangle_{\mathcal{A}}|^2 d\mu(\omega) \\
&\geq \int_\sigma \int_{\Omega_i} |\langle f, \Lambda^*_{\omega_i} F_i(x)\rangle_{\mathcal{A}}|^2 d\mu(x) d\mu(\omega) + \int_{\sigma^c} \int_{\Omega_i} |\langle f, \Theta^*_{\omega_i} G_i(x)\rangle_{\mathcal{A}}|^2 d\mu(x) d\mu(\omega) \\
&\geq C\langle f, f\rangle_{\mathcal{A}},
\end{aligned}$$

where $B = \sup\left\{B_{F_{\omega_i}}, B_{G_{\omega_i}}\right\}$. Thus, we obtain that $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ and $\{\Theta_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ are c-g-woven for H with universal frame bounds $\dfrac{C}{B}$ and $\dfrac{D}{A}$.

$(II) \Rightarrow (I)$. Suppose that $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ and $\{\Theta_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ are c-g-woven for $\mathcal{H}$ with universal frame bounds C and D. Now, we can write for each $f \in \mathcal{H}$,

$$\begin{aligned}
&\int_{\sigma}\int_{\Omega_i} |\langle f, \Lambda_{\omega_i}^* F_i(x)\rangle_{\mathcal{A}}|^2 d\mu(x)d\mu(\omega) + \int_{\sigma^c}\int_{\Omega_i} |\langle f, \Theta_{\omega_i}^* G_i(x)\rangle_{\mathcal{A}}|^2 d\mu(x)d\mu(\omega)\\
&\quad = \int_{\sigma}\int_{\Omega_i} |\langle \Lambda_{\omega_i} f, F_i(x)\rangle_{\mathcal{A}}|^2 d\mu(x)d\mu(\omega) + \int_{\sigma^c}\int_{\Omega_i} |\langle \Theta_{\omega_i} f, G_i(x)\rangle_{\mathcal{A}}|^2 d\mu(x)d\mu(\omega)\\
&\quad \geq \int_{\sigma} A_{F_{\omega_i}} \langle \Lambda_{\omega_i} f, \Lambda_{\omega_i} f\rangle_{\mathcal{A}} d\mu(\omega) + \int_{\sigma^c} A_{G_{\omega_i}} \langle \Theta_{\omega_i} f, \Theta_{\omega_i} f\rangle_{\mathcal{A}} d\mu(\omega)\\
&\quad \geq A\left(\int_{\sigma} \langle \Lambda_{\omega_i} f, \Lambda_{\omega_i} f\rangle_{\mathcal{A}} d\mu(\omega) + \int_{\sigma^c} \langle \Theta_{\omega_i} f, \Theta_{\omega_i} f\rangle_{\mathcal{A}} d\mu(\omega)\right)\\
&\quad \geq AC\langle f, f\rangle_{\mathcal{A}}.
\end{aligned}$$

Also, we can get

$$\int_{\sigma}\int_{\Omega_i} \langle f, \Lambda_{\omega_i}^* F_i(x)\rangle_{\mathcal{A}} d\mu(x)d\mu(\omega) + \int_{\sigma^c}\int_{\Omega_i} \langle f, \Theta_{\omega_i}^* G_i(x)\rangle_{\mathcal{A}} d\mu(x)d\mu(\omega) \leq BD\langle f, f\rangle_{\mathcal{A}}$$

So, $\left\{\Lambda_{\omega_i}^* F_i\right\}_{\omega\in\Omega, i\in[m]}$ and $\left\{\Theta_{\omega_i}^* G_i\right\}_{\omega\in\Omega, i\in[m]}$ are c-woven for $\mathcal{H}$, with universal bounds AC and BD. □

Theorem 0.13. *Let $\{\Lambda_{\omega_i}\}_{\omega\in\Omega}$ be a c-g-frame in Hilbert C^*-module $\mathcal{H}$ with frame bounds A_i and B_i for each $i \in [m]$. Suppose that there exists $M > 0$ such that for all $f \in \mathcal{H}, i \neq k \in [m]$ and all measurable subset $\Delta \subset \Omega$*

$$\left\|\int_{\Delta} \langle(\Lambda_{\omega_i} - \Lambda_{\omega_k}) f, (\Lambda_{\omega_i} - \Lambda_{\omega_k}) f\rangle_{\mathcal{A}}\, d\mu(\omega)\right\| \leq M \min\left\{\left\|\int_{\Delta} \langle \Lambda_{\omega_i} f, \Lambda_{\omega_i} f\rangle_{\mathcal{A}}\, d\mu(\omega)\right\|, \left\|\int_{\Delta} \langle \Lambda_{\omega_k} f, \Lambda_{\omega_k} f\rangle_{\mathcal{A}}\, d\mu(\omega)\right\|\right\}.$$

Then, the family $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ is a $c-g$-woven with universal bounds

$$\frac{A}{(m-1)(M+1)+1} \quad \text{and} \quad B$$

where, $A := \sum_{i\in[m]} A_i$ and $B = \sum_{i\in[m]} B_i$.

Proof. The upper bound is evident. For the lower bound, suppose that $\{\sigma_i\}_{i\in[m]}$ is a partition of Ω and $f \in \mathcal{H}$. Therefore,

$$\begin{aligned}
\left\|\sum_{i\in[m]} A_i\langle f,f\rangle_{\mathcal{A}}\right\| &\leq \left\|\sum_{i\in[m]}\int_{\Omega}\langle\Lambda_{\omega_i}f,\Lambda_{\omega_i}f\rangle_{\mathcal{A}}d\mu(\omega)\right\| \\
&= \left\|\sum_{i\in[m]}\sum_{k\in[m]}\int_{\sigma_k}\langle\Lambda_{\omega_i}f,\Lambda_{\omega_i}f\rangle_{\mathcal{A}}d\mu(\omega)\right\| \\
&\leq \left\|\sum_{i\in[m]}\left(\int_{\sigma_i}\langle\Lambda_{\omega_i}f,\Lambda_{\omega_i}f\rangle_{\mathcal{A}}d\mu(\omega)\right.\right. \\
&\quad \left.\left. + \sum_{k\in[m]}\int_{\sigma_k}\left\{\langle\Lambda_{\omega_i}f-\Lambda_{\omega_k}f,\Lambda_{\omega_i}f-\Lambda_{\omega_k}f\rangle\, d\mu(\omega)+\langle\Lambda_{\omega_k}f,\Lambda_{\omega_k}f\rangle_{\mathcal{A}}d\mu(\omega)\right\}\right)\right\| \\
&\leq \left\|\sum_{i\in[m]}\left(\int_{\sigma_i}\langle\Lambda_{\omega_i}f,\Lambda_{\omega_i}f\rangle_{\mathcal{A}}d\mu(\omega)+\sum_{\substack{k\in[m]\\k\neq i}}\int_{\sigma_k}(M+1)\langle\Lambda_{\omega_k}f,\Lambda_{\omega_k}f\rangle_{\mathcal{A}}d\mu(\omega)\right)\right\| \\
&= \{(m-1)(M+1)+1\}\left\|\sum_{i\in[m]}\int_{\sigma_i}\langle\Lambda_{\omega_i}f,\Lambda_{\omega_i}f\rangle_{\mathcal{A}}d\mu(\omega)\right\|.
\end{aligned}$$

□

3. Perturbation For C-G-Woven

Perturbation of frames has been discussed by Cazassa and Christensen in [6]. For weaving frames, Bemrose and et al. have studied in [4], also Vashisht and Deepshikha presented for continuous case in [25]. We aim to present it for *c*-*g*-woven.

Theorem 0.14. *Suppose for each $i \in [m]$, the family $\{\Lambda_{\omega_i}\}_{\omega\in\Omega}$ is a $c-g$-frame in $\mathcal{H}$ with frame bounds A_i and B_i. Assume that there exist constants λ_i, η_i and $\gamma_i (i \in [m])$ such that for some fixed $n \in$ [m],*

$$A := A_n - \sum_{i\in[m]\setminus[n]}\left(\lambda_i+\eta_i\sqrt{B_n}+\gamma_i\sqrt{B_i}\right)\left(\sqrt{B_n}+\sqrt{B_i}\right) > 0$$

and

$$\left\|\int_{\Omega}\left\langle\left(\Lambda^*_{\omega_n}-\Lambda^*_{\omega_i}\right)F(w),g\right\rangle_{\mathcal{A}}d\mu\right\| \leq \eta_i\left\|\int_{\Omega}\left\langle\Lambda^*_{\omega_n}F(w),g\right\rangle_{\mathcal{A}}d\mu\right\|+\gamma_i\left\|\int_{\Omega}\left\langle\Lambda^*_{\omega_i}F(w),g\right\rangle_{\mathcal{A}}d\mu\right\|+\lambda_i\|F\|$$

for every $F \in l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega})$ and $g \in \mathcal{H}$. Then for any partition $\{\sigma_j\}_{j\in[m]}$ of Ω, $\{\Lambda_{\omega_i}\}_{\omega\in\sigma_j, j\in[m]}$ is a c-g-frame in Hilbert C^*-Module $\mathcal{H}$ with universal frame bounds A and $\sum_{i\in[m]} B_i$. Hence the family of c-g-frame $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ is woven in Hilbert C^*-Module $\mathcal{H}$.

Proof. It is clear that $\{\Lambda_{\omega_i} f\}_{\omega\in\sigma_j, j\in[m]}$ is strongly measurable for each $f \in \mathcal{H}$ and any partition $\{\sigma_j\}_{j\in[m]}$ of Ω, also the family $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ is a c-g-Bessel family with Bessel bound $\sum_{i\in[m]} B_i$. Now, we show that $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ has the lower frame condition. Assume that T_{Λ_i} is the synthesis operator of $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$. Then for any $F \in l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega})$ for any, $i \in [m]\backslash\{n\}$, we have

$$\begin{aligned}
\|(T_{\Lambda_n} - T_{\Lambda_i}) F\| &= \sup_{\|g\|=1} \left\| \langle (T_{\Lambda_n} - T_{\Lambda_i}) F(w), g\rangle_{\mathcal{A}} \right\| \\
&= \sup_{\|g\|=1} \left\| \langle T_{\Lambda_n} F(w) - T_{\Lambda_i} F(w), g\rangle_{\mathcal{A}} \right\| \\
&= \sup_{\|g\|=1} \left\| \langle T_{\Lambda_n} F(w), g\rangle_{\mathcal{A}} - \langle T_{\Lambda_i} F(w), g\rangle_{\mathcal{A}} \right\| \\
&= \sup_{\|g\|=1} \left\| \left\langle F(w), T^*_{\Lambda_n} g\right\rangle_{\mathcal{A}} - \left\langle F(w), T^*_{\Lambda_i} g\right\rangle_{\mathcal{A}} \right\| \\
&= \sup_{\|g\|=1} \left\| \langle F(w), \Lambda_{\omega_n} g\rangle_{\mathcal{A}} - \langle F(w), \Lambda_{\omega_i} g\rangle_{\mathcal{A}} \right\| \\
&= \sup_{\|g\|=1} \left\| \left\langle \Lambda^*_{\omega_n} F(w), g\right\rangle_{\mathcal{A}} - \left\langle \Lambda^*_{\omega_i} F(w), g\right\rangle_{\mathcal{A}} \right\| \\
&= \sup_{\|g\|=1} \left\| \int_\Omega \left\langle \left(\Lambda^*_{\omega_n} - \Lambda^*_{\omega_i}\right) F(w), g\right\rangle_{\mathcal{A}} d\mu \right\| \\
&\le \eta_i \sup_{\|g\|=1} \left\| \int_\Omega \left\langle \Lambda^*_{\omega_n} F(w), g\right\rangle_{\mathcal{A}} d\mu \right\| + \gamma_i \sup_{\|z\|=1} \left\| \int_\Omega \left\langle \Lambda^*_{\omega_i} F(w), g\right\rangle_{\mathcal{A}} d\mu \right\| \\
&\quad + \lambda_i \|F\| \\
&= \eta_i \|T_{\Lambda_n} F\| + \gamma_i \|T_{\Lambda_i} F\| + \lambda_i \|F\| \\
&\le \eta_i \sqrt{B_n} \|F\| + \gamma_i \sqrt{B_i} \|F\| + \lambda_i \|F\| \\
&= \left(\eta_i \sqrt{B_n} + \gamma_i \sqrt{B_i} + \lambda_i\right) \|F\|.
\end{aligned}$$

Thus,

$$\|T_{\Lambda_n} - T_{\Lambda_i}\| \le \eta_i \sqrt{B_n} + \gamma_i \sqrt{B_i} + \lambda_i. \tag{3}$$

For each $i \in [m]$ and $\sigma \subset \Omega$, we define

$$\begin{aligned}
T_i^{(\sigma)} &: (\oplus_{\omega\in\sigma} H_\omega, \mu)_{L^2} \to H \\
\left\langle T_i^{(\sigma)} G, h\right\rangle_{\mathcal{A}} &= \int_\sigma \left\langle \Lambda^*_{\omega_i} G(w), h\right\rangle_{\mathcal{A}} d\mu
\end{aligned}$$

for all $G \in l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega})$, we have

$$\begin{aligned} \left\|T_i^{(\sigma)} G\right\| &= \left\|T_{\Lambda_i}\left(G \cdot \chi_\sigma\right)\right\| \\ &\leq \left\|T_{\Lambda_i}\right\| \left\|G_{\cdot\chi_\sigma}\right\| \\ &\leq \left\|T_{\Lambda_i}\right\| \|F\|_2 \\ &\leq \sqrt{B_i}\|F\|_2 \end{aligned}$$

thus, $\left\|T_i^{(\sigma)}\right\| \leq \sqrt{B_i}$, for each $i \in [m]$. Similarly with (3), we get for each $i \in [m]\backslash\{n\}$,

$$\left\|T_n^{(\sigma)} - T_i^{(\sigma)}\right\| \leq \eta_i\sqrt{B_n} + \gamma_i\sqrt{B_i} + \lambda_i.$$

For every $f \in H$ and $i \in [m]\backslash\{n\}$, we compute

$$\begin{aligned} \left\|\left(T_n^{(\sigma)}\left(T_n^{(\sigma)}\right)^* - T_i^{(\sigma)}\left(T_i^{(\sigma)}\right)^*\right) f\right\| \leq& \left\|\left(T_n^{(\sigma)}\left(T_n^{(\sigma)}\right)^* - T_n^{(\sigma)}\left(T_i^{(\sigma)}\right)^*\right) f\right\| \\ &+ \left\|\left(T_i^{(\sigma)}\left(T_i^{(\sigma)}\right)^* - T_n^{(\sigma)}\left(T_i^{(\sigma)}\right)^*\right) f\right\| \\ \leq& \left\|T_n^{(\sigma)}\right\| \left\|\left(\left(T_n^{(\sigma)}\right)^* - \left(T_i^{(\sigma)}\right)^*\right) f\right\| \\ &+ \left\|T_i^{(\sigma)}\right\| \left\|\left(T_n^{(\sigma)} - T_i^{(\sigma)}\right) f\right\| \\ \leq& \left(\eta_i\sqrt{B_n} + \gamma_i\sqrt{B_i} + \lambda_i\right)\left(\sqrt{B_n} + \sqrt{B_i}\right)\|f\|. \end{aligned}$$

Now, suppose that $\{\sigma_i\}_{i\in[m]}$ is a partition of Ω and T_Λ be the synthesis operator associated with the c-g-Bessel family $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$, we have

$$
\begin{aligned}
\|T_\Lambda^* f\|^2 &= \|\langle f, T_\Lambda T_\Lambda^* f\rangle_{\mathcal{A}}\| \\
&= \left\| \sum_{i\in[m]} \int_{\sigma_i} \left\langle f, \Lambda_{\omega_i}^* \Lambda_{\omega_i} f \right\rangle_{\mathcal{A}} d\mu(\omega) \right\| \\
&= \left\| \int_{\sigma_1} \left\langle f, \Lambda_{\omega_1}^* \Lambda_{\omega_1} f \right\rangle_{\mathcal{A}} d\mu(\omega) + \ldots + \int_{\sigma_n} \left\langle f, \Lambda_{\omega_n}^* \Lambda_{\omega_n} f \right\rangle_{\mathcal{A}} d\mu(\omega) \quad + \ldots \right. \\
&\qquad \left. \ldots + \int_{\sigma_m} \left\langle f, \Lambda_{\omega_m}^* \Lambda_{\omega_m} f \right\rangle_{\mathcal{A}} d\mu(\omega) \right\| \\
&= \left\| \int_{\sigma_1} \left\langle f, \Lambda_{\omega_1}^* \Lambda_{\omega_1} f \right\rangle_{\mathcal{A}} d\mu(\omega) + \ldots + \sum_{i\in[m]} \int_{\sigma_i} \left\langle f, \Lambda_{\omega_i}^* \Lambda_{\omega_i} f \right\rangle_{\mathcal{A}} d\mu(\omega) \right. \\
&\qquad \left. - \sum_{i\in[m]\setminus\{n\}} \int_{\sigma_i} \left\langle f, \Lambda_{\omega_i}^* \Lambda_{\omega_i} f \right\rangle_{\mathcal{A}} d\mu(\omega) + \int_{\sigma_m} \left\langle f, \Lambda_{\omega_m}^* \Lambda_{\omega_m} f \right\rangle_{\mathcal{A}} d\mu(\omega) \right\| \\
&= \left\| \int_{\Omega} \left\langle f, \Lambda_{\omega_n}^* \Lambda_{\omega_n} f \right\rangle_{\mathcal{A}} d\mu(\omega) - \right. \\
&\qquad \left. \sum_{i\in[m]\setminus\{n\}} \left(\int_{\sigma_i} \left\langle f, \Lambda_{\omega_n}^* \Lambda_{\omega_n} f \right\rangle_{\mathcal{A}} d\mu(\omega) \; - \int_{\sigma_i} \left\langle f, \Lambda_{\omega_i}^* \Lambda_{\omega_i} f \right\rangle_{\mathcal{A}} d\mu(\omega) \right) \right\| \\
&\geq \left\| \int_{\Omega} \left\langle f, \Lambda_{\omega_n}^* \Lambda_{\omega_n} f \right\rangle_{\mathcal{A}} d\mu(\omega) \right\| - \\
&\qquad \sum_{i\in[m]\setminus(n)} \left\| \int_{\sigma_i} \left\langle f, \Lambda_{\omega_n}^* \Lambda_{\omega_n} f \right\rangle_{\mathcal{A}} d\mu(\omega) - \int_{\sigma_i} \left\langle f, \Lambda_{\omega_i}^* \Lambda_{\omega_i} f \right\rangle_{\mathcal{A}} d\mu(\omega) \right) \right\| \\
&\geq A_n \|\langle f, f\rangle\|_{\mathcal{A}} - \sum_{i\in[m]\setminus\{n\}} \left\| \left\langle T_n^{(\sigma)} \left(T_n^{(\sigma)}\right)^* f, f \right\rangle_{\mathcal{A}} - \left\langle T_i^{(\sigma)} \left(T_i^{(\sigma)}\right)^* f, f \right\rangle_{\mathcal{A}} \right\|
\end{aligned}
$$

$$
\begin{aligned}
&\geq A_n \|\langle f, f\rangle\|_{\mathcal{A}} - \sum_{i\in[m]\setminus\{n\}} \|\langle f, f\rangle\|_{\mathcal{A}} \left\| \left(T_n^{(\sigma)} \left(T_n^{(\sigma)}\right)^* - T_i^{(\sigma)} \left(T_i^{(\sigma)}\right)^* \right) \right\| \\
&\geq A_n \|\langle f, f\rangle\|_{\mathcal{A}} - \sum_{i\in[m]\setminus\{n\}} \|\langle f, f\rangle\|_{\mathcal{A}} \left(\eta_i \sqrt{B_n} + \gamma_i \sqrt{B_i} + \lambda_i \right) \left(\sqrt{B_n} + \sqrt{B_i} \right) \\
&= A \langle f, f\rangle_{\mathcal{A}}.
\end{aligned}
$$

□

Corollary 0.15. *For each $i \in [m]$, let the family $\{\Lambda_{\omega_i}\}_{\omega\in\Omega}$ be a $c-g$-frame in Hilbert C^*-module $\mathcal{H}$ with frame bounds A_i and B_i. Assume that there exist constants $\lambda_i, \eta_i, \gamma_i (i \in$*

$[m-1]$) *and* $n \in [m]$ *so that*

$$A := A_1 - \sum_{i \in [m-1] \setminus \{n\}} \left(\lambda_i + \eta_i \sqrt{B_i} + \gamma_i \sqrt{B_{i+1}}\right)\left(\sqrt{B_i} + \sqrt{B_{i+1}}\right) > 0$$

and

$$\left\| \int_\Omega \left\langle \left(\Lambda^*_{\omega_i} - \Lambda^*_{\omega(i+1)}\right) F(w), g \right\rangle d\mu \right\| \leq \eta_i \left\| \int_\Omega \left\langle \Lambda^*_{\omega_i} F(w), g \right\rangle_{\mathcal{A}} d\mu \right\| + \gamma_i \left\| \int_\Omega \left\langle \Lambda^*_{\omega(i+1)} F(w), g \right\rangle_{\mathcal{A}} d\mu \right\| + \lambda_i \|\langle F, F\rangle_{\mathcal{A}}\|^{\frac{1}{2}},$$

for every $F \in l^2(\Omega, \{\mathcal{H}_w\}_{\omega\in\Omega})$ *and* $g \in \mathcal{H}$. *Then for any partition* $\{\sigma_j\}_{j\in[m]}$ *of* Ω, $\{\Lambda_{\omega_i}\}_{\omega\in\sigma_j, j\in[m]}$ *is a* $c-g$ *-frame in Hilbert* C^**-Module* $\mathcal{H}$ *with universal frame bounds* A *and* $\sum_{i\in[m]} B_i$. *Hence the family of c-g-frame* $\{\Lambda_{\omega_i}\}_{\omega\in\Omega, i\in[m]}$ *is woven in Hilbert* C^**-Module* $\mathcal{H}$.

Declarations

Availablity of data and materials

Not applicable.

Competing interest

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Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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