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Tensor product for g-fusion frame in Hilbert C^*-modules

Fakhr-dine Nhari[1], Mohamed Rossafi[2], Youssef Aribou[3]

[1]Laboratory Analysis, Geometry and Applications Department of Mathematics, Faculty of Sciences, University of Ibn Tofail, Kenitra, Morocco

[2]Department of Mathematics, Faculty of Sciences, Dhar El Mahraz University Sidi Mohamed Ben Abdellah, P. O. Box 1796 Fez Atlas, Morocco

[3]Laayoune Higher School of Technology, Ibn Zohr University, Laayoune, Morocco

E-mail: [1]nharidoc@gmail.com, [2]rossafimohamed@gmail.com, [3]aribouyoussef3@gmail.com

Abstract.

In this paper, we stady the tensor product of $g-$fusion frame in Hilbert C^*-modules and we give the frame operator for a pair of $g-$fusion Bessel sequences in tensor product of Hilbert C^*-modules.

Keywords: G-fusion Frame, tensor product, Hilbert C^*-module.

1. Introduction and Preliminaries

In the study of vector spaces one of the most important concepts is that of a basis, allowing each element in the space to be written as a linear combination of the elements in the basis. However, the conditions to a basis are very restrictive: linear independence between the elements. This makes it hard or even impossible to find bases satisfying extra conditions, and this is the reason that one might look for a more flexible substitute. Frames are such tools. A frame for a vector space equipped with inner product also allows each element in the space to be written as a linear combination of the elements in the frame, but linear independence between the frame elements is not required.

Frames for Hilbert spaces were introduced by Duffin and schaefer [5] in 1952 to study some deep problems in nonharmonic fourier series by abstracting the fondamental notion of Gabor [6] for signal processing.

Many generalizations of the concept of frame have been defined in Hilbert C^*-modules [7, 9, 11, 12, 13, 14, 15]

Tensor products have important applications, for example tensor products are useful in the approximation of multi-variate functions of combinations of univariate ones.

In this paper we consider the tensor product of Hilbert C^*-modules and we generalize some of known results about frames to generalized-fusion frames in Hilbert C^*-modules.

The paper is organized as follows, we continue this introductory section we briefly recall the definitions and basic properties of C^*-algebra and Hilbert C^*-modules. In section 2, we introduce the concept of $g-$fusion frame, and gives an equivalent definition. In section 3, we introduce the concept of the tensor product of $g-$fusion frame and gives some properties. Finally in section 4, we discuss the frame operator for a pair of $g-$fusion Bessel sequences in tensor product of Hilbert C^*-modules.

In the following we briefly recall the definitions and basic properties of C^*-algebra, Hilbert $\mathcal{A}$-modules. Our reference for C^*-algebras is [4, 3]. For a C^*-algebra $\mathcal{A}$ if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1.1. [3]. If $\mathcal{A}$ is a Banach algebra, an involution is a map $a \to a^*$ of $\mathcal{A}$ into itself such that for all a and b in $\mathcal{A}$ and all scalars α the following conditions hold:

(1) $(a^*)^* = a$.

(2) $(ab)^* = b^*a^*$.

(3) $(\alpha a + b)^* = \bar{\alpha}a^* + b^*$.

Definition 1.2. [3]. A C^*-algebra $\mathcal{A}$ is a Banach algebra with involution such that :

$$\|a^*a\| = \|a\|^2$$

for every a in $\mathcal{A}$.

Example 1.3. $\mathcal{B} = B(H)$ the algebra of bounded operators on a Hilbert space, is a C^*-algebra, where for each operator A, A^* is the adjoint of A.

Definition 1.4. [8]. Let $\mathcal{A}$ be a unital C^*-algebra and H be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and U are compatible. H is a pre-Hilbert $\mathcal{A}$-module if H is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : H \times H \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x \rangle \geq 0$ for all $x \in H$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

(ii) $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in H$.

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in H$.

For $x \in H$, we define $||x|| = ||\langle x,x\rangle||^{\frac{1}{2}}$. If H is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$.

Throughout this paper I and J are finite or countably infinite index sets, we fix the notations $\mathcal{A}$ and $\mathcal{B}$ for a given unital C^*-algebras, H and K are the countably generated Hilbert $\mathcal{A}-$module and $\mathcal{B}-$module, respectively. Let $\{H_i\}_{i\in I}$ and $\{K_j\}_{j\in J}$ are the sequences of closed orthogonally complemented submodules of H and K, respectively. $\{W_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ are the sequences of Hilbert C^*-modules. $End^*_{\mathcal{A}}(H,W_i)$ is a set of all adjointable operator from H to W_i. In particular $End^*_{\mathcal{A}}(H)$ denote the set of all adjointable operators on H. P_{H_i} denote the orthogonal projection onto the closed submodule orthogonally complemented H_i of H.

Define the Hilbert $\mathcal{A}-$module

$$l^2(\{W_i\}_{i\in I}) = \{\{x_i\}_{i\in I} : x_i \in W_i, \| \sum_{i\in I}\langle x_i,x_i\rangle\| < \infty\}$$

with $\mathcal{A}-$valued inner product $\langle\{x_i\}_{i\in I},\{y_i\}_{i\in I}\rangle = \sum_{i\in I}\langle x_i,y_i\rangle$, where $\{x_i\}_{i\in I},\{y\}_{i\in I} \in l^2(\{W_i\}_{i\in I})$

Lemma 1.5. [2]. *Let H and K two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(H,K)$. Then the following statements are equivalent:*

(i) *T is surjective.*

(ii) *T^* is bounded below with respect to norm, i.e., there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$ for all $x \in K$.*

(iii) *T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $\langle T^*x, T^*x\rangle \geq m'\langle x,x\rangle$ for all $x \in K$.*

Lemma 1.6. [1]. *Let U and H two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(U,H)$. Then:*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and*

$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2.$$

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

Lemma 1.7. [2] *Let H be a Hilbert $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$, and $T \in End^*_{\mathcal{A}}(H)$ such that $T^* = T$. The following statements are equivalent:*

(i) *T is surjective.*

(ii) *There are $m, M > 0$ such that $m\|x\| \leq \|Tx\| \leq M\|x\|$, for all $x \in H$.*

(iii) *There are $m', M' > 0$ such that $m'\langle x,x\rangle \leq \langle Tx,Tx\rangle \leq M'\langle x,x\rangle$ for all $x \in H$.*

2. $g-$**fusion frame in Hilbert** C^*-**modules**

We begin this section with the following lemma:

Lemma 2.1. *Let* $\{H_i\}_{i\in I}$ *be a sequence of orthogonally complemented closed submodules of* H *and* $T \in End^*_{\mathcal{A}}(H)$ *invertible, if* $T^*TH_i \subseteq H_i$ *for each* $i \in I$*, then* $\{TH_i\}_{i\in I}$ *is a sequence of orthogonally complemented closed submodules and* $P_{H_i}T^* = P_{H_i}T^*P_{TH_i}$*.*

Proof. Firstly for each $i \in I$, $T : H_i \to TH_i$ is invertible, so each TH_i is a closed submodule of H. We show that $H = TH_i \oplus T(H_i^{\perp})$. Since $H = TH$, then for each $x \in H$, there exists $y \in H$ sutch that $x = Ty$. On the other hand $y = u + v$, for some $u \in H_i$ and $v \in H_i^{\perp}$. Hence $x = Tu + Tv$, where $Tu \in TH_i$ and $Tv \in T(H_i^{\perp})$, plainly $TH_i \cap T(H_i^{\perp}) = (0)$, therefore $H = TH_i \oplus T(H_i^{\perp})$. Hence for every $y \in H_i$, $z \in H_i^{\perp}$ we have $T^*Ty \in H_i$ and therefore $\langle Ty, Tz\rangle = \langle T^*Ty, z\rangle = 0$, so $T(H_i^{\perp}) \subset (TH_i)^{\perp}$ and consequently $T(H_i^{\perp}) = (TH_i)^{\perp}$ witch implies that TH_i is orthogonally complemented.
Let $x \in H$ we have $x = P_{TH_i}x+y$, for some $y \in (TH_i)^{\perp}$, then $T^*x = T^*P_{TH_i}x+T^*y$. Let $v \in H_i$ then $\langle T^*y, v\rangle = \langle y, Tv\rangle = 0$ then $T^*y \in H_i^{\perp}$ and we have $P_{H_i}T^*x = P_{H_i}T^*P_{TH_i}x + P_{H_i}T^*y$, then $P_{H_i}T^*x = P_{H_i}T^*P_{TH_i}x$ thus implies that for each $i \in I$ we have $P_{H_i}T^* = P_{H_i}T^*P_{TH_i}$. □

Definition 2.2. Let $\{H_i\}_{i\in I}$ be a sequence of closed orthogonally complemented submodules of H, $\{v_i\}_{i\in I}$ be a familly of positive weights in $\mathcal{A}$, i.e., each v_i is a positive invertible element from the center of the C^*-algebra $\mathcal{A}$ and $\Lambda_i \in End^*_{\mathcal{A}}(H, W_i)$ for all $i \in I$. We say that $\Lambda = \{H_i, \Lambda_i, v_i\}_{i\in I}$ is a $g-$fusion frame for H if and only if there exists two constants $0 < A \leq B < \infty$ such that

$$A\langle x, x\rangle \leq \sum_{i\in I} v_i^2\langle \Lambda_iP_{H_i}x, \Lambda_iP_{H_i}x\rangle \leq B\langle x, x\rangle, \qquad \forall x \in H. \tag{2.1}$$

The constants A and B are called the lower and upper bounds of $g-$fusion frame, respactively. If $A = B$ then Λ is called tight g-fusion frame and if $A = B = 1$ then we say Λ is a Parseval $g-$fusion frame. If Λ satisfies the inequality

$$\sum_{i\in I} v_i^2\langle \Lambda_iP_{H_i}x, \Lambda_iP_{H_i}x\rangle \leq B\langle x, x\rangle, \qquad \forall x \in H.$$

then it is called a $g-$fusion Bessel sequence with bound B in H.

Lemma 2.3. *let* $\Lambda = (H_i, \Lambda_i, v_i)_{i\in I}$ *be a* $g-$*fusion Bessel sequence for* H *with bound* B*. Then for each sequence* $\{x_i\}_{i\in I} \in l^2(\{W_i\}_{i\in I})$*, the series* $\sum_{i\in I} v_iP_{H_i}\Lambda_i^*x_i$ *is converge unconditionally.*

Proof. let J be a finite subset of I, then

$$\begin{aligned}||\sum_{i\in J} v_i P_{H_i}\Lambda_i^* x_i|| &= \sup_{||y||=1} ||\langle \sum_{i\in J} v_i P_{H_i}\Lambda_i^* x_i, y\rangle|| \\ &\leq ||\sum_{i\in J}\langle x_i, x_i\rangle||^{\frac{1}{2}} \sup_{||y||=1} ||\sum_{i\in J} v_i^2 \langle \Lambda_i P_{H_i} y, \Lambda_i P_{H_i} y\rangle||^{\frac{1}{2}} \\ &\leq \sqrt{B}||\sum_{i\in J}\langle x_i, x_i\rangle||^{\frac{1}{2}}.\end{aligned}$$

And it follows that $\sum_{j\in I} v_j P_{W_j}\Lambda_j^* f_j$ is unconditionally convergent in H. □

Now, we can define the synthesis operator by lemma 2.3

Definition 2.4. let $\Lambda = (H_i, \Lambda_i, v_i)_{i\in I}$ be a $g-$fusion Bessel sequence for H. Then the operator $T_\Lambda : l^2(\{W_i\}_{i\in I}) \to H$ defined by

$$T_\Lambda(\{x_i\}_{i\in I}) = \sum_{i\in I} v_i P_{W_i}\Lambda_i^* x_i, \qquad \forall \{x_i\}_{i\in I} \in l^2(\{W_i\}_{i\in I}).$$

Is called synthesis operator. We say the adjoint T_Λ^* of the synthesis operator the analysis operator and it is defined by $T_\Lambda^* : \mathcal{H} \to l^2(\{W_i\}_{i\in I})$ such that

$$T_\Lambda^*(x) = \{v_i \Lambda_i P_{H_i}(x)\}_{i\in I}, \qquad \forall x \in H.$$

The operator $S_\Lambda : H \to H$ defined by

$$S_\Lambda x = T_\Lambda T_\Lambda^* x = \sum_{j\in I} v_i^2 P_{H_i}\Lambda_i^* \Lambda_i P_{H_i}(x), \qquad \forall x \in H.$$

Is called $g-$fusion frame operator. It can be easily verify that

$$\langle S_\Lambda x, x\rangle = \sum_{i\in I} v_i^2 \langle \Lambda_i P_{H_i}(x), \Lambda_i P_{H_i}(x)\rangle, \qquad \forall x \in H. \tag{2.2}$$

Furthermore, if Λ is a $g-$fusion frame with bounds A and B, then

$$A\langle x, x\rangle \leq \langle S_\Lambda x, x\rangle \leq B\langle x, x\rangle, \qquad \forall x \in H.$$

It easy to see that the operator S_Λ is bounded, self-adjoint, positive, now we proof the inversibility of S_Λ. Let $x \in H$ we have

$$||T_\Lambda^*(x)|| = ||\{v_i \Lambda_i P_{W_i}(x)\}_{i\in I}|| = ||\sum_{i\in I} v_i^2 \langle \Lambda_i P_{H_i}(x), \Lambda_i P_{H_i}(x)\rangle||^{\frac{1}{2}}.$$

Since Λ is $g-$fusion frame then

$$\sqrt{A}||\langle x, x\rangle||^{\frac{1}{2}} \leq ||T_\Lambda^* x||.$$

Then

$$\sqrt{A}||x|| \leq ||T_\Lambda^* x||.$$

Frome lemma 1.5, T_Λ is surjective and by lemma 1.6, $T_\Lambda T_\Lambda^* = S_\Lambda$ is invertible. We now, $AI_H \leq S_\Lambda \leq BI_H$ and this gives $B^{-1}I_H \leq S_\Lambda^{-1} \leq A^{-1}I_H$

Theorem 2.5. *Let H be a Hilbert $\mathcal{A}-$module over C^*-algebra. Then $\Lambda = (H_i, \Lambda_i, v_i)_{i\in I}$ is a $g-$fusion frame for H if and only if there exist two constants $0 < A \leq B < \infty$ such that for all $x \in H$*

$$A||\langle x, x\rangle|| \leq ||\sum_{i\in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x\rangle|| \leq B||\langle x, x\rangle||.$$

Proof. Suppose Λ is $g-$fusion frame for H, then for all $x \in H$,

$$A||\langle x, x\rangle|| \leq ||\sum_{i\in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x\rangle|| \leq B||\langle x, x\rangle||$$

Conversely, for each $x \in H$ we have

$$\begin{aligned} ||\sum_{i\in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x\rangle|| &= ||\sum_{i\in I} \langle v_i \Lambda_i P_{H_i} x, v_i \Lambda_i P_{H_i} x\rangle|| \\ &= ||\langle \{v_i \Lambda_i P_{H_i} x\}_{i\in I}, \{v_i \Lambda_i P_{H_i} x\}_{i\in I}\rangle|| \\ &= ||\{v_i \Lambda_i P_{H_i} x\}_{i\in I}||^2. \end{aligned}$$

We define the operator $L : \mathcal{H} \to \oplus_{i\in I} W_i$ by $L(x) = \{v_i \Lambda_i P_{H_i} x\}_{i\in I}$, then

$$||L(x)||^2 = ||(v_i \Lambda_i P_{H_i} x)_{i\in I}||^2 \leq B||x||^2.$$

L is $\mathcal{A}-$linear bounded operator, then there exist $C > 0$ sutch that

$$\langle L(x), L(x)\rangle \leq C\langle x, x\rangle, \qquad \forall x \in \mathcal{H}.$$

So

$$\sum_{j\in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x\rangle \leq C\langle x, x\rangle, \qquad \forall x \in H.$$

Therefore Λ is a $g-$fusion Bessel sequence for $\mathcal{H}$. Now we cant define the $g-$fusion frame operator S_Λ on $\mathcal{H}$. So

$$\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} x, \Lambda_j P_{W_j} x\rangle = \langle S_\Lambda x, x\rangle, \qquad \forall x \in H.$$

Since S_Λ is positive, self-adjoint, then

$$\langle S_\Lambda^{\frac{1}{2}} x, S_\Lambda^{\frac{1}{2}} x\rangle = \langle S_\Lambda x, x\rangle, \qquad \forall x \in H.$$

That implies

$$A||\langle x, x\rangle|| \leq ||\langle S_\Lambda^{\frac{1}{2}} x, S_\Lambda^{\frac{1}{2}} x\rangle|| \leq B||\langle x, x\rangle||, \qquad \forall x \in H.$$

Frome lemma 1.7 there exist two canstants $A', B' > 0$ such that

$$A'\langle x, x\rangle \leq \langle S_\Lambda^{\frac{1}{2}} x, S_\Lambda^{\frac{1}{2}} x\rangle \leq B'\langle x, x\rangle, \qquad \forall f \in H.$$

So

$$A'\langle x, x\rangle \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x\rangle \leq B'\langle x, x\rangle, \qquad \forall x \in H.$$

Hence Λ is a $g-$fusion frame for H. □

3. Tensor product of $g-$fusion frame in Hilbert C^*-modules

Suppose that $\mathcal{A}, \mathcal{B}$ are unitals C^*-algebras and we take $\mathcal{A} \otimes \mathcal{B}$ as the completion of $\mathcal{A} \otimes_{alg} \mathcal{B}$ with the spatial norm. $\mathcal{A} \otimes \mathcal{B}$ is the spatial tensor product of $\mathcal{A}$ and $\mathcal{B}$, also suppose that H is a Hilbert $\mathcal{A}-$module and K is a Hilbert $\mathcal{B}-$module. We want to define $H \otimes K$ as a Hilbert $(\mathcal{A} \otimes \mathcal{B})-$module. Start by forming the algebraic tensor product $H \otimes_{alg} K$ of the vector spaces H, K (over $\mathbb{C}$). This is a left module overh $(\mathcal{A} \otimes_{alg} \mathcal{B})$ (the module action being given by $(a \otimes b)(x \otimes y) = ax \otimes by(a \in \mathcal{A}, b \in \mathcal{B}, x \in H, y \in K))$. For $(x_1, x_2 \in H, y_1, y_2 \in K)$ we define $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{\mathcal{A} \otimes \mathcal{B}} = \langle x_1, x_2 \rangle_{\mathcal{A}} \otimes \langle y_1, y_2 \rangle_{\mathcal{B}}$. We also know that for $z = \sum_{i=1}^{n} x_i \otimes y_i$ in $H \otimes_{alg} K$ we have $\langle z, z \rangle_{\mathcal{A} \otimes \mathcal{B}} = \sum_{i,j} \langle x_i, x_j \rangle_{\mathcal{A}} \otimes \langle y_i, y_j \rangle_{\mathcal{B}} \geq 0$ and $\langle z, z \rangle_{\mathcal{A} \otimes \mathcal{B}} = 0$ iff $z = 0$. This extends by linearity to an $(\mathcal{A} \otimes_{alg} \mathcal{B})-$valued sesquilinear from on $H \otimes_{alg} K$, which makes $H \otimes_{alg} K$ into a semi-inner-product module over the pre-C^*-algebra $(\mathcal{A} \otimes_{alg} \mathcal{B})$. The semi-inner-product on $H \otimes_{alg} K$ is actually an inner product. see [10]. Then $H \otimes_{alg} K$ is an inner-product module over the pre-C^*-algebra $(\mathcal{A} \otimes_{alg} \mathcal{B})$, and we can perform the double completion discussed in chapter 1 of [10] to conclude that the completion $H \otimes K$ of $\mathcal{A} \otimes_{alg} \mathcal{B}$ is a Hilbert $(\mathcal{A} \otimes \mathcal{B})-$module. We call $H \otimes K$ the exterior tensor product of H and K. With H, K as above, we wish to investigate the adjointable operators on $H \otimes K$. Suppose that $S \in End^*_{\mathcal{A}}(H)$ and $T \in End^*_{\mathcal{B}}(K)$. We define a linear operator $S \otimes T$ on $H \otimes K$ by $S \otimes T(x \otimes y) = Sx \otimes Ty$ $(x \in H, y \in K)$. It is a routine verification that is $S^* \otimes T^*$ is the adjoint of $S \otimes T$, so in fact $S \otimes T \in End^*_{\mathcal{A} \otimes \mathcal{B}}(H \otimes K)$. We note that if $a \in \mathcal{A}^+$ and $b \in \mathcal{B}^+$, then $a \otimes b \in (\mathcal{A} \otimes \mathcal{B})^+$. Plainly if a, b are Hermitian elements of $\mathcal{A}$ and $a \leq b$, then for every positive element x of $\mathcal{B}$, we have $a \otimes x \geq b \otimes x$.

Definition 3.1. Let $\{v_i\}_{i \in I}$, $\{w_j\}_{j \in J}$ be two families of positive weights, i.e., each v_i and w_j are positive invertible elements of $\mathcal{A}$, and $\Lambda_i \otimes \Gamma_j \in End^*_{\mathcal{A}}(H \otimes K, W_i \otimes V_j)$ for each $i \in I$ and $j \in J$. Then the family $\Lambda \otimes \Gamma = \{H_i \otimes K_j, \Lambda_i \otimes \Gamma_j, v_i w_j\}_{i,j}$ is saide to be a generalized fusion frame or $g-$fusion frame for $H \otimes K$ with respect to $\{H_i \otimes K_j\}_{i,j}$ if there exist constants $0 < A \leq B < \infty$ such that for all $x \otimes y \in H \otimes K$

$$A\langle x \otimes y, x \otimes y \rangle \leq \sum_{i,j} v_i^2 w_j^2 \langle (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j}(x \otimes y), (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j}(x \otimes y) \rangle \leq B\langle x \otimes y, x \otimes y \rangle$$

where $P_{H_i \otimes K_j}$ is the orthogonal projection of $H \otimes K$ onto $H_i \otimes K_j$. The constants A and B are called the frame bounds of $\Lambda \otimes \Gamma$. If $A = B$ then it is called a tight $g-$fusion frame. If the family $\Lambda \otimes \Gamma$ satisfies the inequality, for each $x \otimes y \in H \otimes K$

$$\sum_{i,j} v_i^2 w_j^2 \langle (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j}(x \otimes y), (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j}(x \otimes y) \rangle \leq B\langle x \otimes y, x \otimes y \rangle$$

then it is called a $g-$fusion Bessel sequence in $H \otimes K$ with bound B.

Definition 3.2. For $i \in I$ and $j \in J$, define the Hilbert $\mathcal{A} \otimes \mathcal{B}-$module

$$l^2(\{W_i \otimes V_j\}) = \{\{x_i \otimes y_j\} : x_i \otimes y_j \in W_i \otimes V_j, \| \sum_{i,j} \langle x_i \otimes y_j, x_i \otimes y_j \rangle \| < \infty\}$$

with the $\mathcal{A} \otimes \mathcal{B}-$valued inner product

$$\begin{aligned}
\langle \{x_i \otimes y_j\}, \{x_i' \otimes y_j'\} \rangle &= \sum_{i,j} \langle x_i \otimes y_j, x_i' \otimes y_j' \rangle \\
&= \sum_{i,j} \langle x_i, x_i' \rangle_{W_i} \otimes \langle y_j, y_j' \rangle_{V_j} \\
&= \left(\sum_{i \in I} \langle x_i, x_i' \rangle_{W_i} \right) \otimes \left(\sum_{j \in J} \langle y_j, y_j' \rangle_{V_j} \right) \\
&= \langle \{x_i\}_{i \in I}, \{x_i'\}_{i \in I} \rangle_{l^2(\{W_i\}_{i \in I})} \otimes \langle \{y_j\}_{j \in J}, \{y_j\}_{j \in J} \rangle_{l^2(\{V_j\}_{j \in J})}.
\end{aligned}$$

Theorem 3.3. *The families $\Lambda = \{H_i, \Lambda_i, v_i\}_{i \in I}$ and $\Gamma = \{K_j, \Gamma_j, w_j\}_{j \in J}$ are $g-$fusion frames for H and K with respect to $\{W_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$, respectively if and only if the family $\Lambda \otimes \Gamma = \{H_i \otimes K_j, \Lambda_i \otimes \Gamma_j, v_i w_j\}_{i,j}$ is a $g-$fusion frame for $H \otimes K$ with respect $\{W_i \otimes V_j\}_{i,j}$.*

Proof. Suppose that Λ and Γ are $g-$fusion frame for H and K. Then there exist positive constants (A, B) and (C, D) such that

$$A\|x\|^2 \leq \| \sum_{i \in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x \rangle_{\mathcal{A}} \| \leq B\|x\|^2, \forall x \in H,$$

$$C\|y\|^2 \leq \| \sum_{j \in J} w_j^2 \langle \Gamma_j P_{K_j} y, \Gamma_j P_{K_j} y \rangle_{\mathcal{B}} \| \leq D\|y\|^2, \forall y \in K,$$

then,

$$AC\|x\|^2\|y\|^2 \leq \| \sum_{i \in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x \rangle_{\mathcal{A}} \| \| \sum_{j \in J} w_j^2 \langle \Gamma_j P_{K_j} y, \Gamma_j P_{K_j} y \rangle_{\mathcal{B}} \| \leq BD\|x\|^2\|y\|^2,$$

hence,

$$AC\|x \otimes y\|^2 \leq \| \sum_{i \in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x \rangle_{\mathcal{A}} \otimes \sum_{j \in J} w_j^2 \langle \Gamma_j P_{K_j} y, \Gamma_j P_{K_j} y \rangle_{\mathcal{B}} \| \leq BD\|x \otimes y\|^2,$$

so,

$$AC\|x \otimes y\|^2 \leq \| \sum_{i,j} v_i^2 w_j^2 \langle \Lambda_i P_{H_i} x \otimes \Gamma_j P_{K_j} y, \Lambda_i P_{H_i} x \otimes \Gamma_j P_{K_j} y \rangle \| \leq BD\|x \otimes y\|^2.$$

Therefore, for each $x \otimes y \in H \otimes K$

$$AC\|x \otimes y\|^2 \leq \| \sum_{i,j} v_i^2 w_j^2 \langle (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j} (x \otimes y), (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j} (x \otimes y) \rangle \| \leq BD\|x \otimes y\|^2,$$

we conclude that, $\Lambda \otimes \Gamma$ is a $g-$fusion frame for $H \otimes K$. Conversely, suppose that $\Lambda \otimes \Gamma$ is a $g-$fusion frame for $H \otimes K$, then there exist constants $A, B > 0$ such that for all $x \otimes y \in H \otimes K - \{0 \otimes 0\}$,

$$A\|x \otimes y\|^2 \leq \| \sum_{i,j} v_i^2 w_j^2 \langle (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j}(x \otimes y), (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j}(x \otimes y) \rangle \| \leq B\|x \otimes y\|^2,$$

then,

$$A\|x \otimes y\|^2 \leq \| \sum_{i \in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x \rangle_{\mathcal{A}} \otimes \sum_{j \in J} w_j^2 \langle \Gamma_j P_{K_j} y, \Gamma_j P_{K_j} y \rangle_{\mathcal{B}} \| \leq B\|x \otimes y\|^2,$$

hence,

$$A\|x\|^2\|y\|^2 \leq \| \sum_{i \in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x \rangle_{\mathcal{A}} \| \| \sum_{j \in J} w_j^2 \langle \Gamma_j P_{K_j} y, \Gamma_j P_{K_j} y \rangle_{\mathcal{B}} \| \leq B\|x\|^2\|y\|^2,$$

So,

$$\frac{A\|y\|^2}{\| \sum_{j \in J} w_j^2 \langle \Gamma_j P_{K_j} y, \Gamma_j P_{K_j} y \rangle_{\mathcal{B}} \|} \|x\|^2 \leq \| \sum_{i \in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x \rangle_{\mathcal{A}} \|,$$

and

$$\| \sum_{i \in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x \rangle_{\mathcal{A}} \| \leq \frac{B\|y\|^2}{\| \sum_{j \in J} w_j^2 \langle \Gamma_j P_{K_j} y, \Gamma_j P_{K_j} y \rangle_{\mathcal{B}} \|} \|x\|^2.$$

Therefore, Λ is a $g-$fusion frame for H. Similarly, it can be schown that Γ is a $g-$fusion frame for K. □

Definition 3.4. Let $\Lambda \otimes \Gamma = \{H_i \otimes K_j, \Lambda_i \otimes \Gamma_j, v_i w_j\}_{i,j}$ be a $g-$fusion frame for $H \otimes K$. The synthesis operator $T_{\Lambda\Gamma} : l^2(\{W_i \otimes V_j\}) \to H \otimes K$ is given by

$$T_{\Lambda\Gamma}(\{x_i \otimes y_j\}) = \sum_{i,j} v_i w_j P_{H_i \otimes K_j} (\Lambda_i \otimes \Gamma_j)^*(x_i \otimes y_j), \quad \forall \{x_i \otimes y_j\} \in l^2(\{W_i \otimes V_j\}).$$

And the frame operator $S_{\Lambda\otimes\Gamma} : H \otimes K \to H \otimes K$ is described by

$$S_{\Lambda\otimes\Gamma}(x \otimes y) = \sum_{i,j} (v_i w_j)^2 P_{H_i \otimes K_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j}(x \otimes y), \quad \forall x \otimes y \in H \otimes K.$$

Theorem 3.5. *If S_Λ, S_Γ and $S_{\Lambda\otimes\Gamma}$ are the associated $g-$fusion frame operators and T_Λ, T_Γ and $T_{\Lambda\otimes\Gamma}$ are the synthesis operators of $g-$fusion frames Λ, Γ and $\Lambda \otimes \Gamma$ for H, K and $H \otimes K$, respectively, then $S_{\Lambda\otimes\Gamma} = S_\Lambda \otimes S_\Gamma$, $S_{\Lambda\otimes\Gamma}^{-1} = S_\Lambda^{-1} \otimes S_\Gamma^{-1}$, and , $T_{\Lambda\otimes\Gamma} = T_\Lambda \otimes T_\Gamma$, $T_{\Lambda\otimes\Gamma}^* = T_\Lambda^* \otimes T_\Gamma^*$.*

Proof. Let each $x \otimes y \in H \otimes K$, we have

$$\begin{aligned}
S_{\Lambda\otimes\Gamma}(x\otimes y) &= \sum_{i,j} v_i^2 w_j^2 P_{H_i\otimes K_j}(\Lambda_i\otimes\Gamma_j)^*(\Lambda_i\otimes\Gamma_j)P_{H_i\otimes K_j}(x\otimes y)\\
&= \sum_{i,j} v_i^2 w_j^2 (P_{H_i\otimes K_j})(\Lambda_i\otimes\Gamma_j)^*(\Lambda_i\otimes\Gamma_j)(P_{H_i}\otimes P_{K_j})(x\otimes y)\\
&= \sum_{i,j} v_i^2 w_j^2 (P_{H_i}\Lambda_i^*\Lambda_i P_{H_i}x)\otimes(P_{K_j}\Gamma_j^*\Gamma_j P_{K_j}y)\\
&= \left(\sum_{i\in I} v_i^2 P_{H_i}\Lambda_i^*\Lambda_i P_{H_i}x\right)\otimes\left(\sum_{j\in J} w_j^2 P_{K_j}\Gamma_j^*\Gamma_j P_{K_j}y\right)\\
&= S_\Lambda x\otimes S_\Gamma y\\
&= (S_\Lambda\otimes S_\Gamma)(x\otimes y).
\end{aligned}$$

Then, $S_{\Lambda\otimes\Gamma} = S_\Lambda \otimes S_\Gamma$, so $S^{-1}_{\Lambda\otimes\Gamma} = S_\Lambda^{-1}\otimes S_\Gamma^{-1}$.

On the other hand, for each $\{x_i\otimes y_j\}\in l^2(\{W_i\otimes V_j\})$, we have

$$\begin{aligned}
T_{\Lambda\otimes\Gamma}(\{x_i\otimes y_j\}) &= \sum_{i,j} v_i w_j P_{H_i\otimes K_j}(\Lambda_i\otimes\Gamma_j)^*(x_i\otimes y_j)\\
&= \left(\sum_{i\in I} v_i P_{H_i}\Lambda_i^* x_i\right)\otimes\left(\sum_{j\in J} w_j P_{K_j}\Gamma_j^* y_j\right)\\
&= T_\Lambda(\{x_i\})\otimes T_\Gamma(\{y_j\})\\
&= (T_\Lambda\otimes T_\Gamma)(\{x_i\otimes y_j\}).
\end{aligned}$$

this shows that $T_{\Lambda\otimes\Gamma} = T_\Lambda\otimes T_\Gamma$, hence $T^*_{\Lambda\otimes\Gamma} = T^*_\Lambda\otimes T^*_\Gamma$. □

Theorem 3.6. *Let $\Lambda = \{H_i, \Lambda_i, v_i\}_{i\in I}$ and $\Gamma = \{K_j, \Gamma_j, w_j\}_{j\in J}$ be $g-$fusion frames for H and K with $g-$fusion frame operators S_Λ and S_Γ, respectively. Then $\theta = \{S^{-1}_{\Lambda\otimes\Gamma}(H_i\otimes K_j), (\Lambda_i\otimes\Gamma_j)P_{H_i\otimes K_j}S^{-1}_{\Lambda\otimes\Gamma}, v_iw_j\}_{i,j}$ is a $g-$fusion frame for $H\otimes K$.*

Proof. Let (A,B) and (C,D) be the $g-$fusion frame bounds of Λ and Γ, respectively. Now, we have for each $x\otimes y\in H\otimes K$,

$$
\begin{aligned}
&\|\sum_{i,j} v_i^2 w_j^2 \langle (\Lambda_i\otimes\Gamma_j)P_{H_i\otimes K_j}S^{-1}_{\Lambda\otimes\Gamma}P_{S^{-1}_{\Lambda\otimes\Gamma}(H_i\otimes K_j)}(x\otimes y),(\Lambda_i\otimes\Gamma_j)P_{H_i\otimes K_j}S^{-1}_{\Lambda\otimes\Gamma}P_{S^{-1}_{\Lambda\otimes\Gamma}(H_i\otimes K_j)}(x\otimes y)\rangle\|\\
&=\|\sum_{i,j} v_i^2 w_j^2\langle \Lambda_i\otimes\Gamma_j P_{H_i\otimes K_j}S^{-1}_{\Lambda}\otimes S^{-1}_{\Gamma}P_{S^{-1}_{\Lambda}H_i}\otimes P_{S^{-1}_{\Gamma}K_j}(x\otimes y),\\
&\qquad\qquad \Lambda_i\otimes\Gamma_j P_{H_i\otimes K_j}S^{-1}_{\Lambda}\otimes S^{-1}_{\Gamma}P_{S^{-1}_{\Lambda}H_i}\otimes P_{S^{-1}_{\Gamma}K_j}(x\otimes y)\rangle\|\\
&=\|\sum_{i,j} v_i^2 w_j^2\langle \Lambda_i\otimes\Gamma_j P_{H_i}\otimes P_{K_j}S^{-1}_{\Lambda}P_{S^{-1}_{\Lambda}H_i}\otimes S^{-1}_{\Gamma}P_{S^{-1}_{\Gamma}K_j}(x\otimes y),\\
&\qquad\qquad \Lambda_i\otimes\Gamma_j P_{H_i}\otimes P_{K_j}S^{-1}_{\Lambda}P_{S^{-1}_{\Lambda}H_i}\otimes S^{-1}_{\Gamma}P_{S^{-1}_{\Gamma}K_j}(x\otimes y)\rangle\|\\
&=\|\sum_{i,j} v_i^2 w_j^2\langle \Lambda_i P_{H_i}S^{-1}_{\Lambda}P_{S^{-1}_{\Lambda}H_i}x\otimes\Gamma_j P_{K_j}S^{-1}_{\Gamma}P_{S^{-1}_{\Gamma}K_j}y,\Lambda_i P_{H_i}S^{-1}_{\Lambda}P_{S^{-1}_{\Lambda}H_i}x\otimes\Gamma_j P_{K_j}S^{-1}_{\Gamma}P_{S^{-1}_{\Gamma}K_j}y\rangle\|\\
&=\|\sum_{i,j} v_i^2 w_j^2\langle \Lambda_i P_{H_i}S^{-1}_{\Lambda}x\otimes\Gamma_j P_{K_j}S^{-1}_{\Gamma}y,\Lambda_i P_{H_i}S^{-1}_{\Lambda}x\otimes\Gamma_j P_{K_j}S^{-1}_{\Gamma}y\rangle\|\\
&=\|\sum_{i,j} v_i^2 w_j^2\langle \Lambda_i P_{H_i}S^{-1}_{\Lambda}x,\Lambda_i P_{H_i}S^{-1}_{\Lambda}x\rangle_{\mathcal{A}}\otimes\langle\Gamma_j P_{K_j}S^{-1}_{\Gamma}y,\Gamma_j P_{K_j}S^{-1}_{\Gamma}y\rangle_{\mathcal{B}}\|\\
&=\|\sum_{i\in I} v_i^2\langle \Lambda_i P_{H_i}S^{-1}_{\Lambda}x,\Lambda_i P_{H_i}S^{-1}_{\Lambda}x\rangle_{\mathcal{A}}\|\|\sum_{i\in J} w_j^2\langle\Gamma_j P_{K_j}S^{-1}_{\Gamma}y,\Gamma_j P_{K_j}S^{-1}_{\Gamma}y\rangle_{\mathcal{B}}\|\\
&\leq B\|S^{-1}_{\Lambda}x\|^2 D\|S^{-1}_{\Gamma}y\|^2\\
&\leq \frac{BD}{(AC)^2}\|x\otimes y\|^2.
\end{aligned}
$$

On the other hand for each $x\otimes y\in H\otimes K$,

$$
\begin{aligned}
\|x\otimes y\|^4 &= \|\langle x,x\rangle_{\mathcal{A}}\|^2\|\langle y,y\rangle_{\mathcal{B}}\|^2\\
&=\|\langle\sum_{i\in I} v_i^2 P_{H_i}\Lambda_i^*\Lambda_i P_{H_i}S^{-1}_{\Lambda}x,x\rangle_{\mathcal{A}}\|^2\|\langle\sum_{i\in J} w_j^2 P_{K_j}\Gamma_j^*\Gamma_j P_{K_j}S^{-1}_{\Gamma}y,y\rangle_{\mathcal{B}}\|^2\\
&=\|\sum_{i\in I} v_i^2\langle\Lambda_i P_{H_i}S^{-1}_{\Lambda}x,\Lambda_i P_{H_i}x\rangle_{\mathcal{A}}\|^2\|\sum_{j\in J} w_j^2\langle\Gamma_j P_{K_j}S^{-1}y,\Gamma_j P_{K_j}y\rangle_{\mathcal{B}}\|^2\\
&\leq \|\sum_{i\in I} v_i^2\langle\Lambda_i P_{H_i}S^{-1}_{\Lambda}x,\Lambda_i P_{H_i}S^{-1}_{\Lambda}x\rangle_{\mathcal{A}}\|\|\sum_{i\in I} v_i^2\langle\Lambda_i P_{H_i}x,\Lambda_i P_{H_i}x\rangle_{\mathcal{A}}\|\\
&\qquad \|\sum_{j\in J} w_j^2\langle\Gamma_j P_{K_j}S^{-1}_{\Gamma}y,\Gamma_j P_{K_j}S^{-1}_{\Gamma}y\rangle_{\mathcal{B}}\|\|\sum_{j\in J} w_j^2\langle\Gamma_j P_{K_j}y,\Gamma_j P_{K_j}y\rangle_{\mathcal{B}}\|\\
&\leq BD\|x\|^2\|y\|^2\|\sum_{i\in I} v_i^2\langle\Lambda_i P_{H_i}S^{-1}_{\Lambda}P_{S^{-1}_{\Lambda}H_i}x,\Lambda_i P_{H_i}S^{-1}_{\Lambda}P_{S^{-1}_{\Lambda}H_i}x\rangle_{\mathcal{A}}\|\\
&\qquad\qquad \|\sum_{j\in J} w_j^2\langle\Gamma_j P_{K_j}S^{-1}_{\Gamma}P_{S^{-1}_{\Gamma}K_j}y,\Gamma_j P_{K_j}S^{-1}_{\Gamma}P_{S^{-1}_{\Gamma}K_j}y\rangle_{\mathcal{B}}\|\\
&= BD\|x\otimes y\|^2\|\sum_{i,j}(v_i w_j)^2\langle(\Lambda_i\otimes\Gamma_j)P_{H_i\otimes K_j}S^{-1}_{\Lambda\otimes\Gamma}P_{S^{-1}_{\Lambda\otimes\Gamma}(H_i\otimes K_j)}(x\otimes y)\|,
\end{aligned}
$$

hence,

$$\frac{1}{BD}\|x\otimes y\|^2 \leq \|\sum_{i,j}(v_iw_j)^2\langle(\Lambda_i\otimes\Gamma_j)P_{H_i\otimes K_j}S^{-1}_{\Lambda\otimes\Gamma}P_{S^{-1}_{\Lambda\otimes\Gamma}(H_i\otimes K_j)}(x\otimes y)\|.$$

Therefore, θ is a $g-$fusion frame for $H\otimes K$. □

Proposition 3.7. *For the $g-$fusion frame θ, frame operator is $S^{-1}_{\Lambda\otimes\Gamma}$.*

Proof. We put $G=(\Lambda_i\otimes\Gamma_j)P_{H_i\otimes K_j}S^{-1}_{\Lambda\otimes\Gamma}$, then

$$\begin{aligned}G^*G &= ((\Lambda_i\otimes\Gamma_j)P_{H_i\otimes K_j}S^{-1}_{\Lambda\otimes\Gamma})^*(\Lambda_i\otimes\Gamma_j)P_{H_i\otimes K_j}S^{-1}_{\Lambda\otimes\Gamma}\\ &= (S^{-1}_\Lambda\otimes S^{-1}_\Gamma)(P_{H_i}\otimes P_{K_j})(\Lambda_i^*\otimes\Gamma_j^*)(\Lambda_i\otimes\Gamma_j)(P_{H_i}\otimes P_{K_j})(S^{-1}_\Lambda\otimes S^{-1}_\Gamma)\\ &= S^{-1}_\Lambda P_{H_i}\Lambda_i^*\Lambda_iP_{H_i}S^{-1}_\Lambda\otimes S^{-1}_\Gamma P_{K_j}\Gamma_j^*\Gamma_jP_{K_j}S^{-1}_\Gamma\end{aligned}$$

hence,

$$\begin{aligned}&P_{S^{-1}_{\Lambda\otimes\Gamma}(H_i\otimes K_j)}G^*GP_{S^{-1}_{\Lambda\otimes\Gamma}(H_i\otimes K_j)}\\ &= \big(P_{S^{-1}_\Lambda H_i}\otimes P_{S^{-1}_\Gamma K_j}\big)G^*G\big(P_{S^{-1}_\Lambda H_i}\otimes P_{S^{-1}_\Gamma K_j}\big)\\ &= P_{S^{-1}_\Lambda H_i}S^{-1}_\Lambda P_{H_i}\Lambda_i^*\Lambda_iP_{H_i}S^{-1}_\Lambda P_{S^{-1}_\Lambda H_i}\otimes P_{S^{-1}_\Gamma K_j}S^{-1}_\Gamma P_{K_j}\Gamma_j^*\Gamma_jP_{K_j}S^{-1}_\Gamma P_{S^{-1}_\Gamma K_j}\\ &= (P_{H_i}S^{-1}_\Lambda)^*\Lambda_i^*\Lambda_iP_{H_i}S^{-1}_\Lambda\otimes(P_{K_j}S^{-1}_\Gamma)^*\Gamma_j^*\Gamma_jP_{K_j}S^{-1}_\Gamma\\ &= S^{-1}_\Lambda P_{H_i}\Lambda_i^*\Lambda_iP_{H_i}S^{-1}_\Lambda\otimes S^{-1}_\Gamma P_{K_j}\Gamma_j^*\Gamma_jP_{K_j}S^{-1}_\Gamma.\end{aligned}$$

So, for each $x\otimes y\in H\otimes K$, we have

$$\begin{aligned}&\sum_{i,j}v_i^2w_j^2P_{S^{-1}_{\Lambda\otimes\Gamma}(H_i\otimes K_j)}G^*GP_{S^{-1}_{\Lambda\otimes\Gamma}(H_i\otimes K_j)}(x\otimes y)\\ &= \sum_{i,j}v_i^2w_j^2\big(S^{-1}_\Lambda P_{H_i}\Lambda_i^*\Lambda_iP_{H_i}S^{-1}_\Lambda\otimes S^{-1}_\Gamma P_{K_j}\Gamma_j^*\Gamma_jP_{K_j}S^{-1}_\Gamma\big)(x\otimes y)\\ &= \left(\sum_{i\in I}v_i^2S^{-1}_\Lambda P_{H_i}\Lambda_i^*\Lambda_iP_{H_i}S^{-1}_\Lambda x\right)\otimes\left(\sum_{j\in J}w_j^2S^{-1}_\Gamma P_{K_j}\Gamma_j^*\Gamma_jP_{K_j}S^{-1}_\Gamma y\right)\\ &= S^{-1}_\Lambda S_\Lambda(S^{-1}_\Lambda x)\otimes S^{-1}_\Gamma S_\Gamma(S^{-1}_\Gamma y)\\ &= S^{-1}_\Lambda x\otimes S^{-1}_\Gamma y\\ &= (S^{-1}_\Lambda\otimes S^{-1}_\Gamma)(x\otimes y)\\ &= S^{-1}_{\Lambda\otimes\Gamma}(x\otimes y),\end{aligned}$$

we conclude that the corresponding $g-$fusion frame for θ is $S^{-1}_{\Lambda\otimes\Gamma}$. □

Theorem 3.8. *Let $\Lambda=\{H_i,\Lambda_i,v_i\}_{i\in I}$, $\Lambda'=\{H_i',\Lambda_i',v_i'\}_{i\in I}$ be $g-$fusion Bessel sequences with bounds B,D, respectively in H and $\Gamma=\{K_j,\Gamma_j,w_j\}_{j\in J}$, $\Gamma'=\{K_j',\Gamma_j',w_j'\}_{j\in J}$ be $g-$fusion Bessel sequence with bounds E,F, respectively in K. Suppose $(T_\Lambda,T_{\Lambda'})$ and $(T_\Gamma,T_{\Gamma'})$ are their* *synthesis operators such that* $T_{\Lambda'}T_{\Lambda}^* = I_H$ *and* $T_{\Gamma}T_{\Gamma'}^* = I_K$. *Then* $\Lambda \otimes \Gamma = \{H_i \otimes K_j, \Lambda_i \otimes \Gamma_j, v_i w_j\}_{i,j}$ *and* $\Lambda' \otimes \Gamma' = \{H_i' \otimes K_j', \Lambda_i' \otimes \Gamma_j', v_i' w_j'\}_{i,j}$ *are* $g-$*fusion frames for* $H \otimes K$.

Proof. By theorem 3.3, $\Lambda \otimes \Gamma$ and $\Lambda' \otimes \Gamma'$ are $g-$fusion Bessel sequences, respectively in $H \otimes K$. Now, for each $x \otimes y \in H \otimes K$,

$$\begin{aligned}
\|x \otimes y\|^4 &= \|\langle x, x\rangle_{\mathcal{A}}\|^2 \|\langle y, y\rangle_{\mathcal{B}}\|^2 \\
&= \|\langle T_{\Lambda}^* x, T_{\Lambda'}^* x\rangle_{\mathcal{A}}\|^2 \|\langle T_{\Gamma}^* y, T_{\Gamma'}^* y\rangle_{\mathcal{B}}\|^2 \\
&\leq \|T_{\Lambda}^* x\|^2 \|T_{\Lambda'}^* x\|^2 \|T_{\Gamma}^* y\|^2 \|T_{\Gamma'}^* y\|^2 \\
&\leq DF \|x\|^2 \|y\|^2 \|\sum_{i \in I} v_i^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x\rangle_{\mathcal{A}}\| \|\sum_{j \in J} w_j^2 \langle \Gamma_j P_{K_j} y, \Gamma_j P_{K_j} y\rangle_{\mathcal{B}}\| \\
&= DF \|x \otimes y\|^2 \|\sum_{i,j} v_i^2 w_j^2 \langle \Lambda_i P_{H_i} x, \Lambda_i P_{H_i} x\rangle_{\mathcal{A}} \otimes \langle \Gamma_j P_{K_j} y, \Gamma_j P_{K_j} y\rangle_{\mathcal{B}}\| \\
&= DF \|x \otimes y\|^2 \|\sum_{i,j} v_i^2 w_j^2 \langle (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j}(x \otimes y), (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j}(x \otimes y)\rangle\|,
\end{aligned}$$

then,

$$\frac{1}{DF}\|x \otimes y\|^2 \leq \|\sum_{i,j} v_i^2 w_j^2 \langle (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j}(x \otimes y), (\Lambda_i \otimes \Gamma_j) P_{H_i \otimes K_j}(x \otimes y)\rangle\|.$$

Therefore, $\Lambda \otimes \Gamma$ is a $g-$fusion frame for $H \otimes K$. Similarly, it can be shown that $\Lambda' \otimes \Gamma'$ is also a $g-$fusion frame for $H \otimes K$. □

4. Frame operator for a pair of $g-$fusion Bessel sequences in tensor product of Hilbert C^*-modules

Definition 4.1. Let $\Lambda \otimes \Gamma = \{H_i \otimes K_j, \Lambda_i \otimes \Gamma_j, v_i w_j\}_{i,j}$ and $\Lambda' \otimes \Gamma' = \{H_i' \otimes K_j', \Lambda_i' \otimes \Gamma_j', v_i' w_j'\}_{i,j}$ be two $g-$fusion Bessel sequences in $H \otimes K$. Then the operator $S : H \otimes K \to H \otimes K$ defined by for all $x \otimes y \in H \otimes K$,

$$S(x \otimes y) = \sum_{i,j} v_i w_j v_i' w_j' P_{H_i \otimes K_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda_i' \otimes \Gamma_j') P_{H_i' \otimes K_j'}(x \otimes y)$$

is called the frame operator for the pair of $g-$fusion Bessel sequences $\Lambda \otimes \Gamma$ and $\Lambda' \otimes \Gamma'$.

Theorem 4.2. *Let* $S_{\Lambda\Lambda'}$ *and* $S_{\Gamma\Gamma'}$ *be the frame operators for the pair of* $g-$*fusion Bessel sequences* $\left(\Lambda = \{H_i, \Lambda_i, v_i\}_{i \in I}, \Lambda' = \{H_i', \Lambda_i', v_i'\}_{j \in J}\right)$ *and* $\left(\Gamma = \{K_j, \Gamma_j, w_j\}_{j \in J},\right.$ $\left.\Gamma' = \{K_j', \Gamma_j', w_j'\}_{j \in J}\right)$ *in* H *and* K, *respectively. Then* $S = S_{\Lambda\Lambda'} \otimes S_{\Gamma\Gamma'}$.

Proof. We have S is the associated frame operator for the pair of $g-$fusion Bessel sequences $\Lambda \otimes \Gamma$ and $\Lambda^{'} \otimes \Gamma^{'}$, for all $x \otimes y \in H \otimes K$,

$$\begin{aligned}
S(x \otimes y) &= \sum_{i,j} v_i w_j v_i^{'} w_j^{'} P_{H_i \otimes K_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda_i^{'} \otimes \Gamma_j^{'}) P_{H_i^{'} \otimes K_j^{'}} (x \otimes y) \\
&= \sum_{i,j} v_i w_j v_i^{'} w_j^{'} (P_{H_i} \otimes P_{K_j})(\Lambda_i^* \otimes \Gamma_j^*)(\Lambda_i^{'} \otimes \Gamma_j^{'})(P_{H_i^{'}} \otimes P_{K_j^{'}})(x \otimes y) \\
&= \Big(\sum_{i \in I} v_i v_i^{'} P_{H_i} \Lambda_i^* \Lambda_i^{'} P_{H_i^{'}} x \Big) \otimes \Big(\sum_{j \in J} w_j w_j^{'} P_{K_j} \Gamma_j^* \Gamma_j^{'} P_{K_j^{'}} y \Big) \\
&= S_{\Lambda\Lambda^{'}} x \otimes S_{\Gamma\Gamma^{'}} y \\
&= (S_{\Lambda\Lambda^{'}} \otimes S_{\Gamma\Gamma^{'}})(x \otimes y)
\end{aligned}$$

□

Theorem 4.3. *The frame operator for the pair of $g-$fusion Bessel sequences in $H \otimes K$ is bounded.*

Proof. Let $x \otimes y \in H \otimes K$ and $x_1 \otimes y_1 \in H \otimes K$,

$$\begin{aligned}
&\langle S(x \otimes y), x_1 \otimes y_1 \rangle \\
&= \langle \sum_{i,j} v_i w_j v_i^{'} w_j^{'} P_{H_i \otimes K_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda_i^{'} \otimes \Gamma_j^{'}) P_{H_i^{'} \otimes K_j^{'}} (x \otimes y), x_1 \otimes y_1 \rangle \\
&= \sum_{i,j} v_i w_j v_i^{'} w_j^{'} \langle P_{H_i} \Lambda_i^* \Lambda_i^{'} P_{H_i^{'}} x \otimes P_{K_j} \Gamma_j^* \Gamma_j^{'} P_{K_j^{'}} y, x_1 \otimes y_1 \rangle \\
&= \sum_{i,j} v_i w_j v_i^{'} w_j^{'} \langle P_{H_i} \Lambda_i^* \Lambda_i^{'} P_{H_i^{'}} x, x_1 \rangle_{\mathcal{A}} \otimes \langle P_{K_j} \Gamma_j^* \Gamma_j^{'} P_{K_j^{'}} y, y_1 \rangle_{\mathcal{B}} \\
&= \sum_{i \in I} v_i v_i^{'} \langle P_{H_i} \Lambda_i^* \Lambda_i^{'} P_{H_i^{'}} x, x_1 \rangle_{\mathcal{A}} \otimes \sum_{i \in J} w_j w_j^{'} \langle P_{K_j} \Gamma_j^* \Gamma_j^{'} P_{K_j^{'}} y, y_1 \rangle_{\mathcal{B}},
\end{aligned}$$

then,

$$\begin{aligned}
&\|\langle S(x\otimes y), x_1\otimes y_1\rangle\|\\
&=\|\sum_{i\in I} v_i v_i^{'}\langle P_{H_i}\Lambda_i^*\Lambda_i^{'}P_{H_i^{'}}x, x_1\rangle_{\mathcal{A}}\otimes \sum_{i\in J} w_j w_j^{'}\langle P_{K_j}\Gamma_j^*\Gamma_j^{'}P_{K_j^{'}}y, y_1\rangle_{\mathcal{B}}\|\\
&=\|\sum_{i\in I} v_i v_i^{'}\langle P_{H_i}\Lambda_i^*\Lambda_i^{'}P_{H_i^{'}}x, x_1\rangle_{\mathcal{A}}\|\|\sum_{i\in J} w_j w_j^{'}\langle P_{K_j}\Gamma_j^*\Gamma_j^{'}P_{K_j^{'}}y, y_1\rangle_{\mathcal{B}}\|\\
&\leq \|\sum_{i\in I}(v_i^{'})^2\langle \Lambda_i^{'}P_{H_i^{'}}x, \Lambda_i P_{H_i^{'}}x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}\|\sum_{i\in I} v_i^2\langle \Lambda_i P_{H_i}x, \Lambda_i P_{H_i}x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}\\
&\quad\times\|\sum_{j\in J}(w_j^{'})^2\langle \Gamma_j^{'}P_{K_j^{'}}y, \Gamma_j^{'}P_{K_j^{'}}y\rangle_{\mathcal{B}}\|^{\frac{1}{2}}\|\sum_{j\in J} w_j^2\langle \Gamma_j P_{K_j}y_1, \Gamma_j P_{K_j}y_1\rangle_{\mathcal{B}}\|^{\frac{1}{2}}\\
&=\|\sum_{i\in I}(v_i^{'})^2\langle \Lambda_i^{'}P_{H_i^{'}}x, \Lambda_i^{'}P_{H_i^{'}}x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}\|\sum_{j\in J}(w_j^{'})^2\langle \Gamma_j^{'}P_{K_j^{'}}y, \Gamma_j^{'}P_{K_j^{'}}y\rangle_{\mathcal{B}}\|^{\frac{1}{2}}\\
&\quad\times\|\sum_{i\in I} v_i^2\langle \Lambda_i P_{H_i}x, \Lambda_i P_{H_i}x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}\|\sum_{j\in J} w_j^2\langle \Gamma_j P_{K_j}y_1, \Gamma_j P_{K_j}y_1\rangle_{\mathcal{B}}\|^{\frac{1}{2}}\\
&=\|\sum_{i,j}(v_i^2)^{'}(w_j^2)^{'}\langle(\Lambda_i^{'}\otimes\Gamma_j^{'})P_{H_i^{'}\otimes K_j^{'}}(x\otimes y), (\Lambda_i^{'}\otimes\Gamma_j^{'})P_{H_i^{'}\otimes K_j^{'}}(x\otimes y)\rangle\|^{\frac{1}{2}}\\
&\quad\times\|\sum_{i,j} v_i^2 w_j^2\langle(\Lambda_i\otimes\Gamma_j)P_{H_i\otimes K_j}(x_1\otimes y_1), (\Lambda_i\otimes\Gamma_j)P_{H_i\otimes K_j}(x_1\otimes y_1)\rangle\|^{\frac{1}{2}}\\
&\leq\sqrt{B_1B_2}\|x\otimes y\|\otimes\|x_1\otimes y_1\|.
\end{aligned}$$

Let $S_{\Lambda\Lambda'}$ and $S_{\Gamma\Gamma'}$ be the frame operators for the pair of $g-$fusion Bessel sequences $(\Lambda, \Lambda^{'})$ and $(\Gamma, \Gamma^{'})$, respectively. Then by above calculation

$$\begin{aligned}
\|\langle S(x\otimes y), x_1\otimes y_1\rangle\| &= \|\langle(S_{\Lambda\Lambda'}\otimes S_{\Gamma\Gamma'})(x\otimes y), x_1\otimes y_1\rangle\|\\
&=\|\langle S_{\Lambda\Lambda'}x, x_1\rangle_{\mathcal{A}}\otimes\langle S_{\Gamma\Gamma'}y, y_1\rangle_{\mathcal{B}}\|\\
&\leq\sqrt{B_1B_2}\|x\|\|y\|\|x_1\|\|y_1\|,
\end{aligned}$$

so,

$$\sup_{||y_1||=1}\|\langle S_{\Gamma\Gamma'}y, y_1\rangle_{\mathcal{B}}\|\sup_{||x_1||=1}\|\langle S_{\Lambda\Lambda'}x, x_1\rangle_{\mathcal{A}}\|\leq\sqrt{B_1B_2}\|x\|\|y\|$$

hence,

$$\|S_{\Lambda\Lambda'}x\|\|S_{\Gamma\Gamma'}y\|\leq\sqrt{B_1B_2}\|x\|\|y\|$$

do,

$$\frac{\|S_{\Lambda\Lambda'}x\|}{\|x\|}\frac{\|S_{\Gamma\Gamma'}y\|}{\|y\|}\leq\sqrt{B_1B_2},$$

again taking supremum on both side with respect to $\|x\|=1$ and $\|y\|=1$,

$$\|S\|=\|S_{\Lambda\Lambda'}\otimes S_{\Gamma\Gamma'}\|=\|S_{\Lambda\Lambda'}\|\|S_{\Gamma\Gamma'}\|\leq\sqrt{B_1B_2}.$$

□

Declarations

Availablity of data and materials

Not applicable.

Competing interest

The authors declare that they have no competing interests.

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Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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