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Stability Analysis And Maximum Profit Of Logistic Population Model With Time Delay And Constant Effort Of Harvesting

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Abstract

In this paper we develop the logistic population model by considering a time delay and constant effort of harvesting. The time delay makes the model more accurate and harvesting is incorporated since the population is beneficial or the population is under control. We study the sufficient conditions to assure the existence of the population. Perturbation method is used to linearize the model and the stability of the equilibrium point is determined by inspection of the eigenvalues. The results show that there exists a globally asymptotically stable equilibrium point for the model with and without time delay and harvesting. The time delay can induce instability and a Hopf bifurcation can occur. The stable equilibrium point for the model with harvesting is then related to profit function problem. We found that there exists a critical value of the effort that maximizes the profit and the equilibrium point also remains stable. This means that the population can exist and give maximum profit although it is harvested with constant effort of harvesting.

Keywords: *Effort of Harvesting, Eigenvalues, Logistic Model, Profit , Switch Stability, Time Delay*.

1. Introduction

The logistic model can be used to model the growth rate of the population, such as human population, animal, fish in the lake, and trees in the forest. The logistic model was used by Schaefer in Agnew (1979) for analysis of Pacific halibut and yellow fin tuna fisheries. He also said that the parameters of the logistic model may be estimated from the known catch versus catch per unit effort data. Haberman (1998) said that in laboratory experiments, for examples, on the growth of yeast in a culture and on the growth of paramecium, indicated good quantitative agreement to logistic curve. Golec and Sathananthan (2003) have analyzed a stochastic logistic population model for a single species and the result showed that the equilibrium point of the model is asymptotically stable.

Fan and Wang (1998) have examined the exploitation of single population modeled by time dependent logistic equation with periodic coefficient. They showed that the time dependent periodic logistic equation has a unique positive periodic solution, which is globally asymptotically stable. They choose the maximum annual sustainable yield as the management objective and investigate the optimal harvesting policies for constant harvesting and periodic harvesting.

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Time delay or time lag is important to the real world modeling because decision is often made based on historical information. It is important to consider in population model since the growth rate of population do not only depend on the population size at time t but also depends on the population size in the past $(t-\tau)$, where τ is a time delay, Haberman (1998). For example, the current rate of change of human population depends upon the population size at $\tau = 9$ months ago. A certain number of eggs of a fish need a certain time (time delay) to hatch and grow to become adult fish.

The logistic model with and without time delay has been studied by many authors. Nicholson in Barnes and Fulford (2002) has conducted single species experiments on the Australian sheep-blowfly population and the results were well approximated by logistic model with time delay. The logistic delay model with a linear delay harvesting term has also been studied by Berezansky et al., (2004). The existence of the positive solution is considered and sufficient conditions for the existence of the solution are presented. In Rasmussen et al., (2003), the generalized logistic equations where the carrying capacity effect is modeled by a distributed delay effect. If the distributed delay is sufficiently large, oscillations can be introduced as long-term attractors in deference to steady states.

In this paper we develop the logistic population model by considering a time delay, constant effort of harvesting, and a time delay in harvesting term. The time delay is considered in the model under assumption that the growth rate of the population does not depend on the current size of population but also on the past size. Harvesting is involved under consideration that the population is valuable and profitable stock. We will analyze the possible influence of time delay on the stability of the equilibrium point of the model and determine the critical value of the effort that maximizes the profit function.

2. Logistic Model

The logistic model, sometimes called the Verhulst model or logistic growth curve, is a model of population growth. The model is continuous in time which is described by the differential equation

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right).$$

The constant *r*, assumed positive, is called the intrinsic growth rate, since the proportional growth rate for small *x* approximately equals *r*. The positive constant *K* is usually referred to as the environmental carrying capacity, i.e., the maximum sustainable population. The population level *K* is also sometimes called the saturation level, since for large populations there are more deaths than births. The solution of the logistic model together with initial condition $x(0) = x_0 > 0$ is

$$x(t) = \frac{x_0 K}{x_0 + (K - x_0)e^{-rt}}.$$

The logistic model has two equilibrium points, i.e., $x = 0$ and $x = K$. The first equilibrium point is not stable while the second equilibrium point is globally asymptotically stable.

3. Delay Logistic Model with Constant Effort of Harvesting

The single delay logistic model is

$$\frac{dx(t)}{dt} = r\,x(t)\bigl(1 - x(t-\tau)/K\bigr), \tag{1}$$

where τ is a time delay and assume to be positive. A positive equilibrium point of the model is *K*. It has been suggested by Hutchinson in Gopalsamy (1992) that model (1) can be used to model the dynamic of single species population growing towards a saturation level *K* with a constant intrinsic growth rate *r*. The term $\bigl(1 - x(t-\tau)/K\bigr)$ denotes a density dependent feedback mechanism which takes τ units of time to respond to changes in the population density represented in model (1) by *x*. The delay logistic model (1) is well known as delay Verhulst equation or Hutchinson equation. The Hutchinson equation was studied in many papers and text-books, see Hale (1977) and Kuang (1993).

The single delayed logistic model with constant effort of harvesting is

$$\frac{dx(t)}{dt} = r\,x(t)\bigl(1 - x(t-\tau)/K\bigr) - E\,x(t), \tag{2}$$

where *E* is an effort of harvesting which is assumed to be a positive constant. In this model the rate of harvesting is proportional to the size of population at an instant time *t*. The equilibrium point for this model is $x(t) = K(r-E)/r = K_*$. In order to get a nonnegative equilibrium point, we assume that $r > E$.

For analyzing the stability of the equilibrium point, we linearize the model around the equilibrium point. Let $u(t) = x(t) - K_*$ and substitute it into the model (2) to get

$$\frac{du(t)}{dt} = (r-E)\bigl(u(t)+K_*\bigr) - \frac{r}{K}\bigl(u(t)+K_*\bigr)\bigl(u(t-\tau)+K_*\bigr).$$

Hence we have

$$\frac{du(t)}{dt} = -(r-E)\,u(t-\tau). \tag{3}$$

The characteristic equation of model (3) is

$$\lambda + (r-E)e^{-\lambda\tau} = 0. \tag{4}$$

Lemma 1:

Let $r > E > 0$ and $\tau > 0$. The roots of the characteristic equation (4) are negative if $\max\{0,\, r - (\tau e)^{-1}\} < E < r$.

Proof:

Let $F(\lambda) = \lambda + (r-E)\,e^{-\lambda\tau}$. We note from (4) that λ cannot be real nonnegative. We shall show that the roots of $F(\lambda)$ are not complex numbers. We have $F'(\lambda) = 1 - (r\tau - E\tau)\,e^{-\lambda\tau}$ and $\lambda_* = \frac{1}{\tau}\ln(r\tau - E\tau)$ is a critical point for $F(\lambda)$. Further, we have $F''(\lambda) = (r-E)\tau^2 e^{-\lambda\tau}$ which is positive. This means that the value of the critical point gives minimum value for $F(\lambda)$. Now, we have $F(\lambda_*) = \frac{1}{\tau}\bigl(\ln(r\tau - E\tau) + 1\bigr)$ which is less than zero if $(r\tau - E\tau) < e^{-1}$ or $r - E < (\tau e)^{-1}$ and then $0 < r - E < (\tau e)^{-1}$. Further we have $E < r$ and $E > r - (\tau e)^{-1}$. Hence we have $\max\{0,\, r - (\tau e)^{-1}\} < E < r$. □

If $F(\lambda_*) > 0$, i.e., $(r\tau - E\tau) > e^{-1}$, this follows that there is no real root for the characteristic equation (4). In this condition the characteristic equation has a complex conjugate root. If we let $\lambda = \rho + i\omega$, $\rho \in R$, $\omega \in [0, \infty)$, as a root for (4), we have

$$\rho + i\omega = -re^{-(\rho+i\omega)\tau} = -re^{-\rho\tau}\left(\cos(\omega\tau) - i\sin(\omega\tau)\right),$$

then we get the two equations for the real and imaginary parts

$$\rho = -(r-E)e^{-\rho\tau}\cos(\omega\tau)\,, \tag{5.a}$$

$$\omega = (r-E)e^{-\rho\tau}\sin(\omega\tau)\,. \tag{5.b}$$

Lemma 2.

Let $r > E > 0$ and $\tau > 0$. The roots of the characteristic equation (4) are complex conjugate with negative real parts if $\max\left\{0,\, r - \pi(2\tau)^{-1}\right\} < E < r - (\tau e)^{-1}$.

Proof:

Let $F(\lambda) = \lambda + (r-E)\,e^{-\lambda\tau}$. We note from (4) that λ cannot be real nonnegative. We have $F'(\lambda) = 1 - (r\tau - E\tau)\,e^{-\lambda\tau}$ and $\lambda_* = \dfrac{1}{\tau}\ln(r\tau - E\tau)$ is a critical point for $F(\lambda)$. Further, we have $F''(\lambda) = (r-E)\tau^2 e^{-\lambda\tau}$ which is positive. This means that the value of the critical point gives minimum value for $F(\lambda)$. The function $F(\lambda)$ has no real roots when $F(\lambda_*) = \left(\ln(r\tau - E\tau) + 1\right)/\tau > 0$ and this occurs when $(r\tau - E\tau) > e^{-1}$ or $E < r - (\tau e)^{-1}$. Now, we shall show that the root of $F(\lambda)$ is a complex number with negative real part. Suppose that (4) has a root $\lambda = \rho + i\omega$ with $\rho \ge 0$. Since $\lambda = 0$ is not a root of characteristic equation (4) we can assume that $\omega > 0$. Since $(r-E)\tau < \pi/2$, then from (5.b) we have $0 < \omega\tau = (r\tau - E\tau)e^{-\rho\tau}\sin\omega\tau < \pi/2$ showing that the left side of equation (5.a), i.e., $\rho = -(r-E)e^{-\rho\tau}\cos\omega\tau$ is nonnegative, while the right side is negative. This contradiction proves that $\rho < 0$. Note that the conjugate of λ also satisfies the characteristic equation (4).□

When the equilibrium point for the model without harvesting is not asymptotically stable and if the population is harvested with constant effort of harvesting and the effort is in the range of $\max\left\{0,\, r - \pi(2\tau)^{-1}\right\} < E < r - (\tau e)^{-1}$, the equilibrium point for the model with harvesting becomes asymptotically stable. In other words, when the equilibrium point for the model without harvesting is not stable, but the population is harvested with constant effort, the population is possibly stable.

Example 1.

Consider model (2) with parameters $r = 2$, $\tau = 1$, and $K = 100$. Take the level of effort of harvesting E as 0 (no harvesting), 0.8, and 0.2. The roots for the related characteristic equation for the various time delays are $0.172816 \pm 1.673686i$, $-0.190462 \pm 1.439223i$, and $0.097214 \pm 1.630353i$ respectively. The trajectories with initial population $x(0) = 80$ for the non linear model are given in figure 1. When the population is not harvested, the population is not stable, but the population becomes possibly stable whenever the population is harvested with constant effort harvesting.

We know that for $\tau = 0$, $\lambda(0) = -(r-E) < 0$; i.e., the zero equilibrium point is asymptotically stable when there is no time delay. Since $\omega = r - E$, then we have

$\omega\tau = \pi/2$. We denote $\tau_0 = \dfrac{\pi}{2\omega} = \dfrac{\pi}{2(r-E)}$. Then the preceding arguments together with the proof of Theorem in Kuang (1993, pp 66) show that when $0 \le \tau < \tau_0$, the zero equilibrium point is asymptotically stable; and when $\tau > \tau_0$, it is unstable. There is a switch stability and Hopf bifurcation occurs at τ_0 and the zero equilibrium point loses stability at this point.

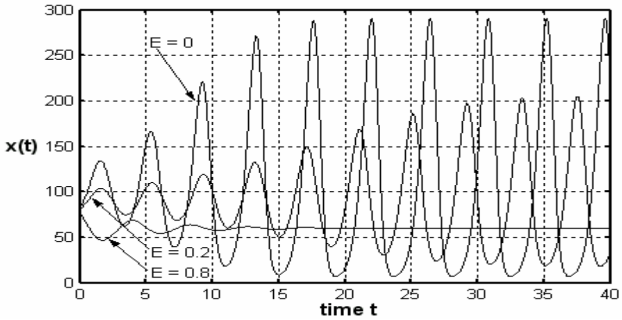

Figure 1: Some trajectories of delay logistic model with several value of *E*.

We now consider condition $E < r$ for model (2) in which the equilibrium point $x = K(r-E)/r$ is locally asymptotically stable for $0 \le \tau < \tau_0$. We would like to relate this equilibrium point to the maximum profit or maximum economic rent problem. We assume that the total cost is proportional to the effort of harvesting. Then the cost function is $TC = c_1 + c_2 E$. The revenue of exploitation, written as total revenue, $TR = p\,E\,x$. The profit function is

$$\pi = TR - TC = pEK\left(\frac{r-E}{r}\right) - c_1 - c_2 E = -\frac{pK}{r}E^2 + (pK - c_2)E - c_1 .$$

From the profit function we have $\dfrac{d\pi}{dE} = -\dfrac{2pKE}{r} + pK - c_2$ and the critical point is $E_c = \dfrac{(pK - c_2)r}{2pK}$. In order to get a positive critical point we assume $pK > c_2$. This assumption has been considered by Clark (1990). The profit maximum occurs at $E = E_c$ since $\dfrac{d^2\pi}{dE^2} = -\dfrac{2pK}{r} < 0$. Consequently, if we choose the effort of harvesting at $E = E_c = \dfrac{(pK - c_2)r}{2pK}$ and the time delay satisfies $0 \le \tau < \tau_0$, then the equilibrium point is stable and also maximizes the profit function.

Example 2.

Consider model (2) with parameters $r = 2$ and $K = 100$. The equilibrium point for the model is $x = 100 - 50E$. Take $c_1 = 1$, $c_2 = 0.5$, and $p = 1$. Then the profit function becomes $\pi = -50E^2 + 99.50E - 1$. Further, we obtain the critical point $E = E_c = 0.9950$. It is easy to see that the profit function is concave down ward. Hence, the critical point $E_c = 0.9950$ gives profit maximum, i.e., $\pi_{\max} = 48.50125$, and the equilibrium point $x = 100 - 50E_c = 50.25$ is also asymptotically stable. The delay margin for stability of the equilibrium point is $\tau_0 = 1.56298$.

4. Logistic Model with Time Delay in Harvesting Term

We now consider the logistic model with constant effort of harvesting where time delay is in the harvesting term. The model is

$$\frac{dx(t)}{dt} = r\,x(t)\big(1 - x(t)/K\big) - E\,x(t-\tau)\ , \qquad (6)$$

where E is an effort of harvesting which is assumed to be a positive constant. In this model the rate of harvesting is proportional to the size of population at time $t-\tau$. Kar (2003) has introduced a time delay in the harvesting term in population dynamics. The equilibrium point for this model is $x(t) = K(r-E)/r = K_*$. In order to get a nonnegative equilibrium point, we assume that $r > E$.

For analyzing the stability of the equilibrium point, we linearize the model around the equilibrium point. Let $u(t) = x(t) - K_*$ and substitute it into the model (6) to get

$$\frac{du(t)}{dt} = r\big(u(t) + K_*\big) - \frac{r}{K}\big(u(t) + K_*\big)^2 - E\big(u(t-\tau) + K_*\big),$$

$$\frac{du(t)}{dt} = ru(t) + rK_* - \frac{r}{K}u(t)^2 - \frac{2r}{K}K_*u(t) - \frac{r}{K}{K_*}^2 - Eu(t-\tau) - EK_* .$$

After neglecting the product terms and simplifying, we obtain

$$\frac{du(t)}{dt} = (2E - r)\,u(t) - E\,u(t-\tau)\ . \qquad (7)$$

The characteristic equation for linear model (7) is

$$\lambda - (2E - r) + Ee^{-\lambda\tau} = 0\ . \qquad (8)$$

Since $r > E$, then $\lambda = 0$ is not a root of the characteristic equation (8).

Theorem 3:

Let $r > E$. The zero equilibrium point of model (7) is asymptotically stable if the following conditions are satisfied;

(i). $\tau < E^{-1}$ and (ii). $\ln(E\tau) - (2E - r)\tau + 1 \le 0$.

Proof:

Let $F(\lambda) = \lambda - (2E - r) + Ee^{-\lambda\tau}$, then we have $F'(\lambda) = 1 - E\tau e^{-\lambda\tau}$ and $\lambda_* = \ln(E\tau)/\tau$ is the critical point for $F(\lambda)$. Since $F''(\lambda) = E\tau^2 e^{-\lambda\tau} > 0$ for any λ, then $F(\lambda)$ is concave upward and $F(\lambda_*) = \dfrac{\ln(E\tau) - (2E - r)\tau + 1}{\tau}$ is minimum. From (i), we have $\lambda_* = \ln(E\tau)/\tau < 0$ and from (ii) we obtain $F(\lambda_*) = \dfrac{\ln(E\tau) - (2E - r)\tau + 1}{\tau} \le 0$.

Note that $F(0) = -(E - r) > 0$ and then $F(\lambda)$ is also positive for some λ, ($\lambda < 0$). Therefore for $\ln(E\tau) - (2E - r)\tau + 1 < 0$, we have λ_2 and λ_1 with property $\lambda_2 < \lambda_* < \lambda_1 < 0$ satisfying $F(\lambda_2) = F(\lambda_1) = 0$. In the case of $\ln(E\tau) - (2E - r)\tau + 1 = 0$, we have only one negative real root, i.e., $\lambda_* = \dfrac{\ln(E\tau)}{\tau}$. This means that the zero equilibrium point of model (7) is asymptotically stable. We also conclude that the equilibrium point $x = K(r - E)/r$ is locally asymptotically stable when the conditions in Theorem 3 are satisfied. □

From the proof of Theorem 3, $F(\lambda) = \lambda - (2E - r) + Ee^{-\lambda\tau}$ is possible to become positive, zero, or negative. It depends on the value of the critical point $\lambda_* = \dfrac{\ln(E\tau)}{\tau}$. When the minimum value of $F(\lambda) = \lambda - (2E - r) + Ee^{-\lambda\tau}$ is positive, this implies there is no real root of it, but complex number will exist.

Theorem 4:

If $E < r/3$ and $\ln(E\tau) - (2E - r)\tau + 1 > 0$, then the roots of characteristic equation (8) are complex conjugate with negative real parts.

Proof:

From the proof of Theorem 3 we have $F(\lambda_*) = \dfrac{\ln(E\tau) - (2E - r)\tau + 1}{\tau}$. Since $\ln(E\tau) - (2E - r)\tau + 1 > 0$, then $F(\lambda_*) > 0$. This means that there is no real root for $F(\lambda) = \lambda - (2E - r) + Ee^{-\lambda\tau}$. Let $\lambda = \rho + i\omega$, $\omega > 0$, is the root of $F(\lambda)$, then we have

$$\rho + i\omega - (2E - r) + Ee^{-\rho\tau}(\cos\omega\tau - i\sin\omega\tau) = 0 .$$

Separating the real and imaginary parts we have

$$\rho - (2E - r) = -Ee^{-\rho\tau}\cos\omega\tau$$
$$\omega = Ee^{-\rho\tau}\sin\omega\tau .$$

We know that there exists a unique $\omega\tau$ in the interval $(0, \pi)$ satisfying both equations. Squaring both equations and adding them yields the equation

$$(\rho - (2E - r))^2 + \omega^2 = E^2 e^{-2\rho\tau} ,$$
$$\rho^2 - 2\rho(2E - r) + (2E - r)^2 + \omega^2 = E^2 e^{-2\rho\tau} .$$

Let $F_1(\rho) = \rho^2 - 2\rho(2E - r) + (2E - r)^2 + \omega^2$ and $F_2(\rho) = E^2 e^{-2\rho\tau}$. Since $r > 3E$ and graphically, we obtain that the intersection between $F_1(\rho)$ and $F_2(\rho)$ occurs for $\rho < 0$. Further, we have complex number $\lambda = \rho + i\omega$ with negative real part. It is easy to see that $\lambda = \rho - i\omega$ is also a root for $F(\lambda)$.□

Theorem 4 follows that if $r > 3E$ and $\ln(E\tau) - (2E - r)\tau + 1 > 0$ then the zero equilibrium point for model (7) is asymptotically stable and the equilibrium point $x = K(r - E)/r$ is locally asymptotically stable.

By Theorem in Kuang (1993, pp 66), we know that if the stability of the trivial solution $u(t) = 0$ of model (7) switches at $\tau = \bar{\tau}$, then the characteristic equation (8) must have a pair of pure conjugate imaginary roots when $\tau = \bar{\tau}$. We can think of the roots of the characteristic equation (8) as continuous functions in terms of the delay τ, i.e., $\lambda(\tau) - (2E - r) + Ee^{-\lambda(\tau)\tau} = 0$.

Therefore, in order to understand the stability switches of model (7), we need to determine the value of $\bar{\tau}$ at which the characteristic equation (8) may have a pair of conjugate pure imaginary roots. We assume $\lambda = i\omega$, $\omega > 0$ is a root of the characteristic equation (8) for $\tau = \bar{\tau}$, $\bar{\tau} \geq 0$. Substituting $\lambda = i\omega$ into the characteristic equation (8), we have:

$$i\omega - (2E - r) + Ee^{-i\omega\tau} = 0 ,$$
$$i\omega - (2E - r) + E\cos\omega\tau - iE\sin\omega\tau = 0 .$$

Separating the real and imaginary parts we get the two equations for the real and imaginary part, i.e.,

$$\begin{aligned}(2E-r) &= E\cos\omega\tau \\ \omega &= E\sin\omega\tau.\end{aligned} \tag{9}$$

Squaring the two equations and adding them together, we obtain

$$\omega^2 = E^2 - (2E-r)^2$$

(10)

If $E^2-(2E-r)^2>0$ or equivalently $E<r<3E$, then we see that purely imaginary roots of the characteristic equation (8) exist.

From equations (9) we have $\cos\omega\tau=(2E-r)/E$ and $\sin\omega\tau=\omega/E$. Hence, there is a unique $\omega\tau$, $0<\omega\tau<2\pi$, such that $\omega\tau$ makes both $\cos\omega\tau=(2E-r)/E$ and $\sin\omega\tau=\omega/E$ hold. Further, we have

$$\tau_1=\theta/\omega, \tag{11}$$

where $0<\theta<2\pi$, $\cot\theta=(2E-r)/\omega$, and ω satisfies (10).

Differentiating the characteristic equation (8) with respect to τ, we have

$$\frac{d\lambda}{d\tau}-Ee^{-\lambda\tau}\left(\tau\frac{d\lambda}{d\tau}+\lambda\right)=0\,.$$

From the characteristic equation (8), we know that $-Ee^{-\lambda\tau}=\lambda-(2E-r)$, hence we have

$$\frac{d\lambda}{d\tau}+\left(\lambda-(2E-r)\right)\left(\tau\frac{d\lambda}{d\tau}+\lambda\right)=0\ ,$$

$$\frac{d\lambda}{d\tau}=\frac{-\lambda^2+\lambda(2E-r)}{1+\lambda\tau-(2E-r)\tau} \tag{12}$$

Thus, the condition $E<r<3E$ implies that purely imaginary roots of the characteristic equation (8) exist. From the equation (12), we have

$$\begin{aligned}\operatorname{sign}\left(\frac{d(\operatorname{Re}\lambda)}{d\tau}\right)_{\lambda=i\omega} &= \operatorname{sign}\left(\operatorname{Re}\frac{d\lambda}{d\tau}\right)_{\lambda=i\omega} = \operatorname{sign}\left(\operatorname{Re}\frac{\omega^2-i\omega(2E-r)}{1+i\omega\tau-(2E-r)\tau}\right) \\ &= \operatorname{sign}\left(\operatorname{Re}\frac{\omega^2+i\omega^3\tau+i\omega(2E-r)(1-(2E-r)\tau)}{\left(1-(2E-r)\tau\right)^2+\omega^2\tau^2}\right) \\ &= \operatorname{sign}\left(\frac{\omega^2}{\left(1-(2E-r)\tau\right)^2+\omega^2\tau^2}\right)\,.\end{aligned}$$

Therefore we see that the sign is always positive. This implies that all the roots that cross the imaginary axis at $i\omega$ cross from left to right as τ increases.

For $\tau=0$, we have $\lambda=E-r<0$ which means the equilibrium point is asymptotically stable. Then the preceding arguments together with the proof of Theorem in Kuang (1993, pp 66) show that when $0\le\tau<\tau_1$, the zero equilibrium point of model (7) is asymptotically stable; and when $\tau>\tau_1$, the zero equilibrium point is unstable. Switch stability occurs at $\tau=\tau_1$ and a Hopf bifurcation occur at this point.

In the case of $r=3E$, we know that $\omega=0$ is the only solution of (10). However, $\lambda=0$ is not the root of the characteristic equation (8) since $r>E$. Hence, there is no stability as well. It is easy to see that if $r>3E$, then there are no pure imaginary roots for the characteristic equation (8). In other words, there are no roots of the characteristic equation (8) crossing the imaginary axis when τ increases. Therefore, there are no stability switches, no matter how the delay τ is chosen.

We now consider condition $E<r$ for model (6) in which the equilibrium point $x=K(r-E)/r$ is globally asymptotically stable for $\tau=0$. We would like to relate this

equilibrium point to the maximum profit. The cost function is $TC = c_1 + c_2 E$, the total revenue is $TR = pEx$, and profit function is:

$$\pi = TR - TC = pEK\left(\frac{r-E}{r}\right) - c_1 - c_2 E = -\frac{pK}{r}E^2 + \left(pK - c_2\right)E - c_1 .$$

The profit function is the same with the profit function in the previous section. We conclude that if we choose the effort of harvesting at $E = E_c = \frac{(pK - c_2)r}{2pK}$, and the time delay satisfies $0 < \tau < \tau_1$, where τ_1 refers to (11), then the equilibrium point is asymptotically stable and also maximizes the profit function.

*Example 3***.**

Consider model (6) with parameters $r = 2$ and $K = 100$. The equilibrium point for the model is $x = 100 - 50E$. Take $c_1 = 1$, $c_2 = 0.5$, and $p = 1$. Then the profit function becomes $\pi = -50E^2 + 99.50E - 1$. Further, we obtain the critical point $E = E_c = 0.9950$. It is easy to see that the profit function is concave down ward. Hence, the critical point $E_c = 0.9950$ gives profit maximum, i.e., $\pi_{\max} = 48.50125$, and the equilibrium point $x = 100 - 50E_c = 50.25$ is also asymptotically stable. The delay margin for stability of the equilibrium point is $\tau_1 = 1.58887$. The Hopf bifurcation occurs at $\tau_1 = 1.58887$. The trajectories around the equilibrium point $x = 50.25$ with various time delays are given in figure 2.

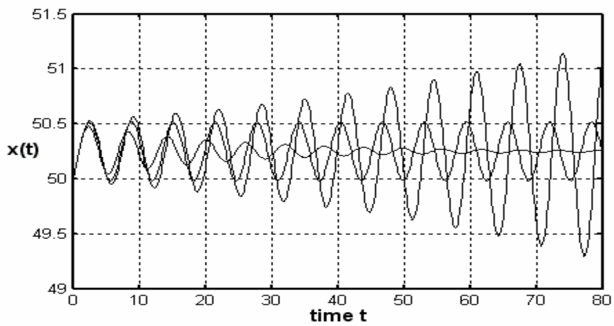

Figure 2: Trajectories of model (6) with $\tau = 1.45; 1.58887; \text{and } 1.65$.

In figure 2 with initial value $x_0 = 50$, the trajectories oscillate around the equilibrium point. For $\tau = 1.45$, the trajectory tends to the equilibrium point, and for $\tau = 1.65$, the trajectory oscillates and diverges. However, for $\tau = 1.58887$, the trajectory oscillates forever and the Hopf bifurcation occurs since if we disturb the value of time delay, the trajectory will converge to the equilibrium point or diverge.

4. Conclusion

In the model without time delay and harvesting, the positive equilibrium point occurs and it is globally asymptotically stable. This means that the population can exist. For the model with time delay and constant effort of harvesting and for the model with time delay in the harvesting term, there exist some conditions for the time delay and effort of harvesting so that the equilibrium point is stable. There exists switch stability and the time delay can induce instability and also a Hopf bifurcation can occur. There exists a critical value of the effort that maximizes the profit function. This means that under suitable value of the parameters, time delay, and effort of harvesting, the population can remain exist and give profit maximum.

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