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Ramsey Number on A Union of Stars Versus A Small Cycle

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Abstract

The Ramsey number for a graph G versus a graph H, denoted by $R(G,H)$, is the smallest positive integer n such that for any graph F of order n, either F contains G as a subgraph or $\bar{F}$ contains H as a subgraph. In this paper, we investigate the Ramsey numbers for stars versus small cycle. We show that $R(S8\,, C4\,) = 10$ and $R(kS_{1+p}\,, C4\,) = k(p+1)+1$ for $k \geq 2$ and $p \geq 3$.

Keywords: Ramsey number, star, cycle.

1. Introduction

Throughout this paper, all graphs are finite and simple. Let G be any graph with the vertex set $V(G)$ and the edge set $E(G)$. The graph G, the *complement* of G, is obtained from the complete graph on $|V\ (G)|$ vertices by deleting the edges of G. A graph $F = (V', E')$ is a subgraph of G if $V' \subseteq V(G)$ and $E' \subseteq E(G)$. For $S \subseteq V(G)$, $G[S]$ represents the *subgraph induced* by S in G. For $v \in V(G)$ and $S \subset V(G)$, the *neighborhood* $N_S(v)$ is the set of vertices in S which are adjacent to v. Furthermore, we define $N_S[v] = N_S(v) \cup \{u\}$. If $S = V\ (G)$, then we use $N(v)$ and $N[v]$ instead of $N_{V(G)}(v)$ and $N_{V(G)}[v]$, respectively. The degree of a vertex v in G is denoted by $d_G(v)$. The order of G, denoted by $|G|$, is the number of its vertices. Let S_n be a *star* on *n* vertices and C_m be a *cycle* on *m* vertices. *Cocktail-party graph* H_s is the graph which is obtained by removing s disjoint edges from K_{2s}. We denote the *complete bipartite* whose partite sets are of order n and p by $K_{n,p}$. A *windmill* graph M_n is a graph on $2n+1$ vertices obtained from n disjoint triangles by identifying precisely one vertex of every triangle.

Given two graphs G and H, the Ramsey number $R(G,H)$ is defined as the smallest natural number n such that for any graph F on n vertices, either F contains G or $\bar{F}$ contains H. Chvatal and Harary (1972) established a useful and general lower bound on the exact Ramsey numbers $R(G,H)$ as follows.

Theorem 1. (Chavatal, Harary, 1972)
Let G and H be teo graphs (not necessarily different) with no isolated vertices. Then the following lower bound holds,

$$R(G,H) \geq (x(G)-1)(n(H)-1)+1,$$

where x(G) is the chromatic number of G and n(H) is the number of vertices in the largest component of H.

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This result of the Chavatal and Harary has motivated various authors to determined the Ramsey numbers R(G,H) for many combinations of graphs G and H, see the nicesurvey paper Radziszowski (2006).

Corollary 1.

$$R(S_{1+p,}C_4) \geq (x(C_4) - 1)(V(S_{1+p}) - 1) + 1 \ = p + 1.$$

Some results about the Ramsey numbers for stars versus cycle have obtained. For instance, Lawrence (1987) showed that $R(S_{16,}C_4)$=20 and

$$R(S_{1+p,}C_4) = \begin{cases} m \; if \; m \geq 2n, \\ 2n + 1 \; if \; m \; is \; odd \; and \; m \leq 2n + 1 \end{cases}.$$

Parsons (1975) considered about the Ramsey numbers for S_{1+p} versus C_4 as presented in Theorem 2.

Theorem 2. (Parson's Upper Bound)

For $p \geq 2,$

$$R(S_{1+p,}C_4) \leq p + \sqrt{p} + 1.$$

Recently, Hasmawati *et al.* (2006, 2009) proved that $R(S_{6,}C_4) = 8$, and $R(S_{6,}K_{2,m}) = 13$ for *m=5* or 6 respectively. Recently, Baskoro *et al.* (2006) determined the Ramsey numbers for multiple copies of a star versus a wheel and for a forest versus a complete graph. Their results are given in the following three theorems.

Theorem 3. (Baskoro *et al.*, 2006)

If m is odd and 5≤m≤2n-1, then

$$R(kS_{n,}W_m) = 3n - 2 + (k - 1)n.$$

Theorem 4. (Baskoro *et al.*, 2006)
For n≥3,

$$R(kS_{n,}W_4) = \begin{cases} (k + 1)n \quad if \; n \; is \; even \; and \; k \geq 2, \\ (k + 1)n - 1 \; if \; n \; is \; odd \; and \; k \geq 1. \end{cases}$$

Theorem 5. (Baskoro *et al.*, 2006)
Let $n_i \geq n_{i+1,}$ *for i = 1,2,...,k-1. If m is such that* $n_i > (n_i - n_{i+1,})(m - 1)$ *for every i,*
then $R(\bigcup_{i=1}^{k} T_{ni,} K_m) = R(T_{nk,}K_m) \ + \ \sum_{i=1}^{k-1} n_{i.}$

In this paper, we study the Ramsey numbers for multiple copies of stars versus small cycle. We determine the Ramsey numbers $R(S_{8,}C_4)$ and $R(kS_{1+p,}C_4)$ for *p≥3* and *k≥2.*

2. Main Results

The results are presented in the next two theorems.

Theorem 6.

$$R(S_{8,}C_4) = 10.$$

Proof. Consider F $\coloneqq H_4 \cup K_{1.}$ Clearly, F has nine vertices and contains no $S_{8.}$ Its complement is isomorphic with M_4. Thus it's clear that M_4 contains no C_4. Hence, we have $R(S_{8,}C_4) \geq 10$. By Parson's upper bound in Parsons (1976), $R(S_{8,}C_4) \leq 8+\sqrt{7}$. Therefore, we have $R(S_{8,}C_4) \leq 10$. Thus, $R(S_{8,}C_4) = 10$.

Lemma 1.
For k≥2 and p≥3. Consider $F \coloneqq K_{k(p+1)-1} \cup K_{1.}$ *F has k(p+1) vertices, however it contains no* $kS_{1+p.}$ *It is easy to see that* $\bar{F}$ *is isomorphic with* $K_{1,k(p+1)-1}$. *So,* $\bar{F}$ *contains no* C_4. *Hence,* $R(kS_{1+p,}C_4) \geq k(p+1)+1$.

Theorem 7.

For p≥3,

$$R(2S_{1+p,}C_4) = 2(p+1)+1.$$

Proof. Let F_1 be a graph of order $2(p+1)+1$ for $p \geq 1$. Suppose $\bar{F}_1$ contains no C_4. By Parsons'supper bound, we have $|F_1| \geq R(S_{1+p,}C_4)$ for $p \geq 1$. Thus $F_1 \supseteq S_{1+p}$. Let $V(S_{1+p}) = \{v_0, v_1, \dots, v_p\}$ with center v_0. Write A=$F_1 \setminus S_{1+p}$ and $T = F_1[A]$. Thus $|T| = p+2$. If there exists $v \in T$ with $d_T(v) \geq p$, then T contains S_{1+p}. Hence F_1 contains $2S_{1+p}$. Therefore, we assume taht for every vertex $v \in T$, $d_T(v) \leq (p-1)$.
Let u be any vertex in T. Write $Q = T \setminus N_T[u]$. Clearly, $|Q| \geq 2$. Observe that if there exists $s \in F_1$ where $s \neq u$ which is not adjacent to at least two vertices in Q, the proof we will use the following assumption.
Assumption 1. Every vertex $s \in F_1$, $s \neq u$ is not adjacent to at most one vertex in Q.
Let u be adjacent to at least $p - |N_T(u)|$ vertices in $S_{1+p} \setminus \{v_0\}$, call them $v_1, \dots, v_p - |N_T(u)|$. Observe that $p - |N_T(u)| = |Q| - 1$. By Assumption 1, vertex v_0 is adjacent to at least $|Q| - 1$ vertices in Q, namely $q_1, \dots, v_p - |N_T(u)|$. Then we have two new stars, namely S'_{1+p} and S''_{1+p}, where
$V(S'_{1+p}) = (S_{1+p} \setminus \{v_1, \dots, v_p - |N_T(u)|\}) \cup \{q_1, \dots, v_p - |N_T(u)|\}$
With v_0 as the center and
$V(S''_{1+p}) = N_T[u] \cup \{v_1, \dots, v_p - |N_T(u)|\}$
With u as the center. Hence, we have $F_1 \supseteq 2S_{1+p}$.
Now we assume that u is adjacent to at most $p - |N_T(u)| - 1$ vertices in $S_{1+p} \setminus \{v_0\}$. This means u is not adjacent to least $|N_T(u)| + 1$ vertices in $S_{1+p} \setminus \{v_0\}$. Let $Y = \{Y = y \in S_{1+p} \setminus \{v_0\} :\ yu \notin E(F_1)\}$. Then $|Y| \geq |N_T(u)| + 1 \geq 1$. It will be shown that there is $y' \in Y$ so that y' is adjacent to all vertices in $N_T(u)$ (see Figure 1). Suppose for every $y \in Y$, there exists $r \in N_T(u)$ such that $yr \notin E(F_1)$. Since $|N_T(u)| < |Y|$, then there exists $r_0 \in N_T(u)$ so that r_0 is not adjacent to at least two vertices in Y, say y_1 and y_2. This implies, $\overline{F_1}[u, r_{0,}y_{1,}y_2]$ forms a C_4, a contradiction. Hence, there exists $y' \in Y$ so that y' is adjacent to all vertices in $N_T(u)$. Furthermore, by Assumption 1 we have that $|N_T(y')| \geq |N_T(u)| + |Q| - 1 = |T| - 2 = p$.

Let q' be the vertex in Q which y'. If $v_0u \notin E(F_1)$, then v_0 must be adjacent to q'. (Otherwise $\bar{F}$ would contain C_4 formed by $\{v_0, y', q', u\}$). Now we have two new stars, namely S^1_{1+p} and S^2_{1+p}, where $VS^1_{1+p} = N_T[y']$ with y' as the center and $VS^2_{1+p} = S_{1+p} \setminus \{y'\} \cup \{q'\}$. If $v_0u \in E(F_1)$, then we also have two new stars. The first one is S^1_{1+p} as in the previous case and the second one is S^3_{1+p} where $VS^3_{1+p} = S_{1+p} \setminus \{y'\} \cup \{u\}$ with v_0 as the center. In case that y' is adjacent with all vertices in Q, then the first star is $N_T[y'] \setminus \{q\}$ and the second star is S^4_{1+p} where $V(S^4_{1+p}) = S_{1+p} \setminus \{y'\} \cup \{q\}$, $q \in Q$ with v_0 as the center. The fact $v_0q \in E(F)$ is guaranteed by Assumption 1. Therefore, we have $R(2S_{1+p,}C_4) = 2(p+1)+1$. The proof is now complete.

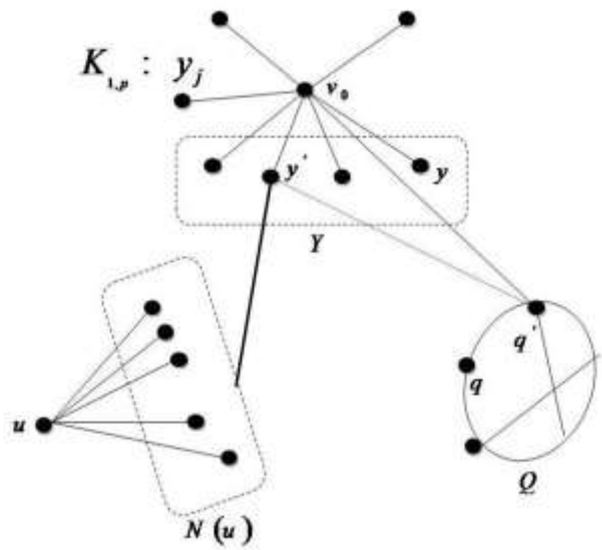

Figure. 1. An illustration of Proof of Theorem 2.

Theorem 3.
For $p \geq 3$ *and* $k \geq 3$,

$$R(kS_{1+p,}C_4) = k(p+1)+1.$$

To obtain the ramsey nmber we use induction on k. We assume the theorem holds for every $2 \leq r \leq k$. Let F_2 be a graph of order *k(p+1)+1*. Suppose $\bar{F}_2 \supseteq kS_{1+p}$. By induction hypothesis, $F_2 \supseteq (k-1)S_{1+p}$. Write $B = F_2 \setminus (k-2)S_{1+p}$ and $T' = F_2[B]$. Thus $|T'| = 2(p+1)+1$. Since $\bar{T}'$contains no C_4 and follows from Theorem 2 that T' contains $2S_{1+p}$. Hence F_2 contains $(k-2)S_{1+p} \cup 2S_{1+p} = kS_{1+p}$. Thus we have $R(kS_{1+p,}C_4) \leq k(p+1)+1$. On the other hand, we have $R(kS_{1+p,}C_4) \geq k(p+1)+1$ (by Lemma 1). The assertion follows.

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