JURNAL MATEMATIKA, STATISTIKA DAN KOMPUTASI e-ISSN: 2614-8811
Published by Departement of Mathematics, Hasanuddin University, Indonesia
https://journal.unhas.ac.id/index.php/jmsk/index
Vol. 21, No. 2, January 2025, pp. 431-438
DOI: 10.20956/j.v21i2.37361

The Chromatic Number of the Edge Corona Operation of Cycle Graph and Star Graph

Bilangan Kromatik Graf Hasil Operasi Korona Sisi Graf Siklus dan Graf Bintang

Alivia Melvina*[1], Kartika Yulianti[2], Yaya S. Kusumah[3]

[1,2,3)]*Mathematics Study Program, Faculty of Mathematics and Science Education, Universitas Pendidikan Indonesia*
***Email**: [1]aliviamelvin@gmail.com, [2]kartika.yulianti@upi.edu, [3]yskusumah@upi.edu*
**Corresponding Author*
Received: 17 September 2024, revised: 20 November 2024, accepted: 29 November 2024

Abstract

One of the concepts in graph theory that can be analyzed is chromatic numbers of a graph and operation of two graphs. There are various kinds of operations of two graphs, one of which is the corona edge operation. This research aims to determine the chromatic number of the edge corona operation of graph $C_n \diamond K_{1,m}$ and $K_{1,m} \diamond C_n$, C_n is a cycle graph and $K_{1,m}$ is a star graph. The chromatic number is determined based on the pattern formed from several n and m values. The results of this research show that the chromatic number of the edge corona operation of graph $C_n \diamond K_{1,m}$ is:

$$\chi(C_n \diamond K_{1,m}) = 4, \quad \text{for } n = 3,4,\dots,k \text{ and } m = 1,2,\dots,l$$

and the chromatic number of the edge corona operation of graph $K_{1,m} \diamond C_n$ is:

$$\chi(K_{1,m} \diamond C_n) = \begin{cases} 5, & \text{if n is an odd number} \\ 4, & \text{if n is an even number} \end{cases}$$

Keywords: Edge Corona Operation, Chromatic Number, Vertex Coloring, Cycle Graph, Star Graph.

Abstrak

Salah satu konsep dalam teori graf yang dapat diteliti adalah bilangan kromatik dari suatu graf dan operasi dari dua graf. Terdapat berbagai macam operasi dari dua graf salah satunya adalah operasi korona sisi. Penelitian ini bertujuan untuk mengetahui bilangan kromatik graf hasil operasi korona sisi graf $C_n \diamond K_{1,m}$ dan $K_{1,m} \diamond C_n$, di mana C_n adalah graf siklus dan $K_{1,m}$ adalah graf bintang. Bilangan kromatik ditentukan berdasarkan pola yang terbentuk dari beberapa nilai n dan m. Dari penelitian ini diperoleh hasil bahwa bilangan kromatik graf hasil operasi korona sisi graf $C_n \diamond K_{1,m}$ adalah:

$$\chi(C_n \diamond K_{1,m}) = 4, \quad \text{untuk } n = 3,4,\dots,k \text{ dan } m = 1,2,\dots,l$$

dan bilangan kromatik graf hasil operasi korona sisi graf $K_{1,m} \diamond C_n$ adalah:

$$(K_{1,m} \diamond C_n) = \begin{cases} 5, & \text{jika } n \text{ adalah bilangan ganjil,} \\ 4, & \text{jika } n \text{ adalah bilangan genap.} \end{cases}$$

Kata kunci: Operasi Korona Sisi, Bilangan Kromatik, Pewarnaan Simpul, Graf Siklus, Graf Bintang

1. INTRODUCTION

Graph theory is one of the fields of applied mathematics that is still developing rapidly. Starting from Leonhard Euler's journey through the problem of the seven Königsberg bridges in 1736 [14]. Through these problems, many concepts from graph theory developed. At the beginning of the 20th century, scientists discovered other benefits of graph theory, for example computer science, transportation, physics, theoretical chemistry, electrical engineering, genetics and others [2].

Graph G is an unordered set $\{V(G), E(G)\}$ where $V(G)$ is a set of non-empty vertices and $E(G)$is a set of edges that connecting a pair of vertices [2, 3]. Two or more graphs will form a new graph operation is carried out [10]. Operations on graphs include *union* ($\cup$), *join* (+), *cartesian product* ($\times$), *composition or lexicographic product* ($[\,]$), and corona ($\odot$) [6]. Hou dan Shiu [7] in their research defined another variation of corona operation. Let G_1 and G_2 be two disjoint graphs n_1, n_2 vertices and m_1, m_2 edges, respectively. The edge corona symbolized by $\diamond$ is a graph operation obtained by taking 1 copy of G_1 and m_1 copies of G_2 then joining the two vertices of the-*i* edge of G_1 to each vertex in the *i* copy of G_2 [6].

Graph coloring is one of the fields in graph theory where certain objects in the graph are given colors which are represented in ordered natural numbers starting from one [1]. Coloring vertices in a graph aims to find out the minimum number of colors provided that there are no identical colors on the vertices that correspond to each other for all the vertices in the graph [17]. The minimum number of colors that needed to color all vertices in a graph is called chromatic number [13]. The chromatic number is denoted by $\chi(G)$ [2, 6, 17].

Many researchers have studied the chromatic number for various graphs. Puspasari, et al., [11] determined the chromatic number of the operation graph and the application scheme of vertex graph coloring. Zakharov [18] found the chromatic numbers of distance graphs G(*n*, 3, 2), for infinitely many *n* and improved the upper bounds for chromatic numbers distance graphs G(*n*, *r*, *s*) for many values of the parameters *r* and *s* and for all sufficiently large *n*. Simanjuntak [13] examined the chromatic numbers of corona operation of cycle graph and cubic graph. Yusuf [17] generalized chromatic numbers in several classes of corona graphs. Recently, Wu, et al [16] studied the chromatic number of heptagraphs and showed that every heptagraph is 3-colorable. Dong, et al [4] examined upper bound of the chromatic number of some P5-free graphs.

In contrast to the previous research, this research will examine chromatic number of the edge corona operation of Cycle Graph and Star Graph. Research related to corona edge operation discussed by Kaspar *et al*. [8], and Liowardani *et al.* [9].

2. LITERATURE REVIEW

In this section, we define edge corona graphs, discuss graph coloring, and present some theorems about chromatic numbers for several graphs. Next, we present the Welch-Powell algorithm for coloring graphs.

2.1 Edge Corona

Definition 2.1. *Edge Corona* [7]. *Let G_1 and G_2 be two graphs on disjoint sets of n_1, n_2 vertices and m_1, m_2 edges, respectively. The edge corona $G_1 \diamond G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and m_1 copies of G_2, and then joining two end-vertices of the i-th edge of G_1 to every vertex in the i-th copy of G_2.*

Based on the definition, the number vertices of corona edge operation graph $\mathrm{G}_1 \diamond \mathrm{G}_2$ is $\mathrm{n}_1 + \mathrm{m}_1\mathrm{n}_2$, while the number of edges of corona edge graph $\mathrm{G}_1 \diamond \mathrm{G}_2$ is $\mathrm{m}_1 + 2\mathrm{m}_1\mathrm{n}_2 + \mathrm{m}_1\mathrm{m}_2$.

2.2 Graph Coloring

Definition 2.2. *Coloring is a mapping of a set of vertices, edges, or regions in a graph G to a set of colors such that no vertices, edges, or regions that adjacent are given the same color* [10].

The following are several theorems used in the proof in the next section.

Theorem 2.1.1 [15] G is a non-null bipartite graph if and only if $\chi(G) = 2$.

Corollary 2.1.2 *If G is a star graph $K_{1,m}$, then $\chi(G) = 2$.*

Theorem 2.1.3 [12] *If C_n is a cycle graph with n vertices, then*

$$\chi(G) = \begin{cases} 2, if\ n\ is\ even \\ 3, if\ n\ is\ odd. \end{cases}$$

2.3 Welch-Powell Algorithm

The Welch-Powell algorithm can be used to color a graph G effectively. The algorithm is as follows [5].

1. Sort the vertices of G in decreasing degree.
2. Use one color to coloring the first vertex that has the highest degree and the other vertex that are not adjacent with the first vertex.
3. Start again with the vertex that has the next highest degree that has not been colored and repeat the process of coloring the vertex using the second color.
4. Repeat adding colors until all the vertices have been colored.

3. METHOD

In this research, the determination of the chromatic number of the edge corona operation $C_n \diamond K_{1,m}$ and $K_{1,m} \diamond C_n$ for several n and m values. This pattern will be generalized for every value of n and m. The steps of this research to obtain the chromatic number are as follows.

1. Review concepts and literature studies.
2. Draw a corona edge operation graph.
3. Examine the vertex chromatic number.
4. Determine the vertex chromatic number.
5. Observe the vertex chromatic number pattern.
6. Determine the vertex chromatic number conjectur.
7. Prove the vertex chromatic number conjectur.
8. Draw conclusions.

4. MAIN RESULTS

In this section, the chromatic number of the corona edge graphs $C_n \diamond K_{1,m}$ and $K_{1,m} \diamond C_n$ for several values of n are presented, then a general proof is carried out.

4.1 $C_n \diamond K_{1,m}$ Graph

Figure 1 displays the coloring of the corona edge graph $C_3 \diamond K_{1,m}$ using the Welch-Powell algorithm. Let $A = \{a_1, a_2, a_3\}$ be the set of vertices of the cycle graph C_3, $B = \{b_i\}$ for $i = 1, 2, 3$ is the set of central vertices of star graph $K_{1,m}$ and $H = \{c_{ij}\}$ for $i = 1, 2, 3$ and $j = 1, 2, \dots, m$ is the set of leaves vertices of star graph $K_{1,m}$.

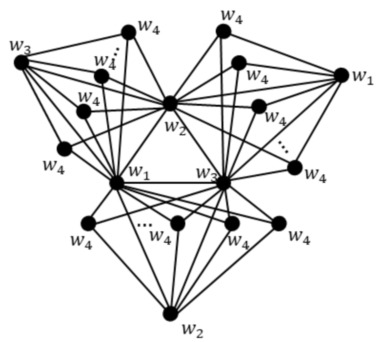

Figure 1. The coloring of $C_3 \diamond K_{1,m}$ graph

Using the same method, we color the corona edge graph $C_n \diamond K_{1,m}$ for several values of n. The same pattern is obtained, the chromatic number is 4. The following is a generalization of the chromatic number of the corona edge of $C_n \diamond K_{1,m}$ graph.

Theorem 1 Let G be the corona edge operation of cycle graph C_n and star graph $K_{1,m}$. The chromatic number of the graph $G = C_n \diamond K_{1,m}$ is

$$\chi(C_n \diamond K_{1,m}) = 4,$$

with $n = 3, 4, \dots, k$ and $m = 1, 2, \dots, l$.

Proof

a) For n as an odd number,

The vertex coloring steps for the $C_n \diamond K_{1,m}$ graph such that $\chi(C_n \diamond K_{1,m}) = 4$ are as follows.

1. The vertices of cycle graph C_n are colored first. Based on Theorem 2.1.3 in Section 2.2, it is known that $\chi(C_n) = 3$ where n is an odd number. This means the minimum number of colors that can be used to color each vertex in a cycle graph C_n is 3 colors. Suppose the color set is $W = \{w_1, w_2, w_3\}$, where w_1 is the color for $a_1, a_3, \dots, a_{n-2}$ vertices, w_2 is the color of $a_2, a_4, \dots, a_{n-1}$ vertices, and w_3 is the color for a_n vertex.
2. Next, coloring for each vertex of star graph $K_{1,m}$. Based on Corollary 2.1.2 it is known that $\chi(K_{1,m}) = 2$. This means that the minimum number of colors that can be used to color the vertices of a star graph $K_{1,m}$ is two colors. Each vertex of a star graph $K_{1,m}$ adjacent with a pair of vertices of the cycle graph C_n, then there is at least one color in W that can be used to color star graph $K_{1,m}$. Therefore, one new color is needed, w_4 so that star graph $K_{1,m}$ can be colored with the minimum number of colors. The following are the coloring steps for the vertices in the star graph $K_{1,m}$.

a. The central vertex of the star graph $K_{1,m}$ which is not adjacent with a_i vertices for $i = 1,3,\dots,n-2$ is colored with w_1. The central vertex of the star graph $K_{1,m}$ which is not adjacent with a_i vertices for $i = 2,4,\dots,n-1$ colored with w_2. The central vertex of the star graph $K_{1,m}$ which is not adjacent with a_n is colored with w_3.
b. Next, assign a color to each leaf vertex of the star graph $K_{1,m}$. Because each leaf vertex adjacent with a central vertex and a pair of vertex of cycle graph C_n, then each leaf vertex of the star graph $K_{1,m}$ colored with a new color, w_4.

So, the minimum number of colors that can be used to color each vertex of the $C_n \diamond K_{1,m}$ graph for odd number n is 4 colors, w_1, w_2, w_3, and w_4, such that $\chi(C_n \diamond K_{1,m}) = 4$.

b) For n as an even number,

The vertex coloring steps for the $C_n \diamond K_{1,m}$ graph such that $\chi(C_n \diamond K_{1,m}) = 4$ are as follows.

1. The vertices of cycle graph C_n are colored first. Based on Theorem 2.1.3 it is known that $\chi(C_n) = 2$ with n is an even number. This means the minimum number of colors that can be used to color each vertex in a cycle graph C_n is 2 colors. Suppose the color set is $W = \{w_1, w_2\}$, where w_1 is the color for $a_1, a_3, \dots, a_{n-1}$ vertices and w_2 is the color of $a_2, a_4, \dots, a_n$ vertices.
2. Next, coloring for each vertex of star graph $K_{1,m}$. Based on Corollary 2.1.2 it is known that $\chi(K_{1,m}) = 2$. This means that the minimum number of colors that can be used to color the vertices of a star graph $K_{1,m}$ is two colors. Each vertex of a star graph $K_{1,m}$ adjacent with a pair of vertices of the cycle graph C_n then 2 colors are needed w_3 and w_4, so that a star graph $K_{1,m}$ can be colored with the minimum number of colors. The following are the coloring steps for the vertices in the star graph $K_{1,m}$.
   a. Each central vertex of star graph $K_{1,m}$ is adjacent with a pair of vertices in the cycle graph C_n, then the central vertex is colored using a new color, w_3.
   b. Next, the coloring for the leaf vertex. Because each leaf vertex is adjacent with a central vertex and a pair of vertices in the cycle graph C_n, the leaves vertex in star graph $K_{1,m}$ are colored with a new color, w_4.

So, the minimum number of colors that can be used to color each vertex of the $C_n \diamond K_{1,m}$ graph for even number n is 4 colors, w_1, w_2, w_3, and w_4, such that $\chi(C_n \diamond K_{1,m}) = 4$.

3.2 $K_{1,m} \diamond C_n$ Graph

Figure 2 displays the coloring of the $K_{1,m} \diamond C_n$ graph using the Welch-Powell algorithm. Let b is a central vertex for star graph $K_{1,m} = \{c_1, c_2, \dots, c_m\}$ be the set of leaves vertex of the star graph $K_{1,m}$, and $A = \{a_{ij}\}$ for $i = 1,2,3$ and $j = 1,2,\dots,m$ is the set of cycle graph C_3.

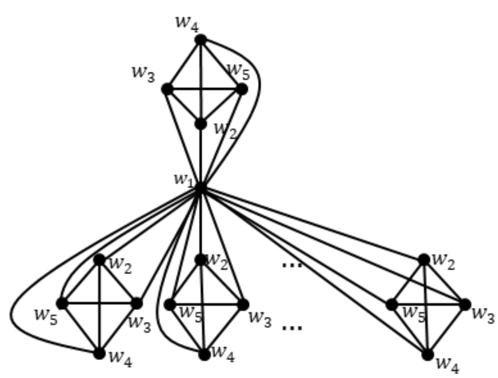

Figure 2. The Coloring of $K_{1,m} \diamond C_n$ Graph

Using the same method, we color $K_{1,m} \diamond C_n$ for several values of n. The same pattern is obtained, the chromatic number is 4 for n is an even number and 5 for n is an odd number. The following is a generalization of the chromatic number of the corona edge of $K_{1,m} \diamond C_n$ graph.

Theorem 2 Let G be the corona edge operation of star graph $K_{1,m}$ and cycle graph C_n. The chromatic number of the graph $G = K_{1,m} \diamond C_n$ is

$$\chi(K_{1,m} \diamond C_n) = \begin{cases} 5, & \text{if } n \text{ is an odd number,} \\ 4, & \text{if } n \text{ is an even number,} \end{cases}$$

where $n = 3,4, \dots, k$ and $m = 1,2, \dots, l$.

Proof. It will be proven that

$$\chi(K_{1,m} \diamond C_n) = \begin{cases} 5, & \text{if } n \text{ is an odd number,} \\ 4, & \text{if } n \text{ is an even number,} \end{cases}$$

a) For n is an odd number,
The steps to coloring every vertex of $K_{1,m} \diamond C_n$ such that $\chi(K_{1,m} \diamond C_n) = 5$ are as follows.
1. The vertices of the star graph $K_{1,m}$ are colored first. Based on Corollary 2.1.2 it is known that $\chi(K_{1,m}) = 2$. This means that the minimum number of colors that can be used to color each vertex of a star graph $K_{1,m}$ is 2 colors. Suppose the color set is $W = \{w_1, w_2\}$, where w_1 is the color for vertex b and w_2 is the color for vertices H.
2. Next, coloring for each vertex of the cycle graph C_n. Based on theorem 2.5.4, for n is an odd number, it is known that $\chi(C_n) = 3$. This means that the minimum number of colors that can be used to color the vertices of cycle graph C_n is 3 colors. Each vertex of a cycle graph $C_n$adjacent with the central vertices and leaf vertices of the star graph $K_{1,m}$, then there are no colors in W that can be used to color the cycle graph C_n. Therefore, 3 new colors are needed, w_3, w_4, w_5 so that $K_{1,m}$ graph can be colored with the minimum number of colors. The following are the coloring steps for each vertex of the cycle graph C_n.
   a. Coloring the vertices of the cycle graph C_n begins by selecting one of the vertices of the cycle graph C_n and giving it the color w_3, example a_1. Other vertices of the cycle graph C_n that are not adjacent with a_1 is $a_3, a_5, \dots a_{n-2}$ can be colored with w_3.
   b. Next, a_2 is adjacent with a_1 and a_3 which are given the color w_3 so they have to be colored with a new color, w_4. Other vertices that are not adjacent with a_2 is $a_4, a_6, \dots, a_{n-1}$ can be colored with w_4.
   c. Next, a_n adjacent with a_1 which is colored with w_3 and a_{n-1} which is colored with w_4 so it must be colored with a new color, w_5.

So, the minimum number of colors that can be used to color each vertex of the $K_{1,m} \diamond C_n$ graph for odd number n is 5 colors, w_1, w_2, w_3, w_4 and w_5, such that $(K_{1,m} \diamond C_n) = 5$.

b) For n is an even number,
1. The vertices of the star graph $K_{1,m}$ are colored first. Based on Corollary 2.1.2, it is known that $\chi(K_{1,m}) = 2$. This means that the minimum number of colors that can be used to color each vertex of a star graph $K_{1,m}$ is 2 colors. Suppose the color set is $W = \{w_1, w_2\}$, where w_1 is the color for vertex b and w_2 is the color for vertices H.
2. Next, coloring for each vertex of the cycle graph C_n. Based on Theorem 2.1.3, for n is an even number, it is known that $\chi(C_n) = 2$. This means that the minimum number of colors

that can be used to color the vertices of cycle graph C_n is 3 colors. Each vertex of a cycle graph $C_n$adjacent with the central vertices and leaf vertices of the star graph $K_{1,m}$, then there are no colors in W that can be used to color the cycle graph C_n. Therefore, 2 new colors are needed, w_3 and w_4 so that $K_{1,m}$ graph can be colored with the minimum number of colors. The following are the coloring steps for each vertex of the cycle graph C_n.

a. Coloring the vertices of the cycle graph C_n begins by selecting one of the vertices of the cycle graph C_n and giving it the color w_3, for example a_1. Other vertices of the cycle graph C_n that are not aligned with a_1 is $a_3, a_5, \dots a_{n-1}$ can be colored with w_3.
b. Next, a_2 is adjacent with a_1 and a_3 which are given the color w_3 so they have to be colored with a new color, w_4. Other vertices that are not adjacent with a_2 is $a_4, a_6, \dots, a_n$ can be colored with w_4.

So, the minimum number of colors that can be used to color each vertex of the $K_{1,m} \diamond C_n$ graph for even number n is 4 colors, w_1, w_2, w_3 and w_4, such that $\left(K_{1,m} \diamond C_n\right) = 4$.

5. CONCLUSION

Based on main results, it can be concluded that:

1. The chromatic number of the edge corona operation of $C_n \diamond K_{1,m}$ is
$$\chi\left(C_n \diamond K_{1,m}\right) = 4, \qquad \text{for } n = 3, 4, \dots, k \text{ and } m = 1, 2, , \dots, l.$$
2. The chromatic number of the edge of corona operation of $K_{1,m} \diamond C_n$ is
$$\chi\left(K_{1,m} \diamond C_n\right) = \begin{cases} 5, & \text{if } n \text{ is an odd number,} \\ 4, & \text{if } n \text{ is an even number.} \end{cases}$$

For further research, edge coloring can be discussed for graphs $C_n \diamond K_{1,m}$ dan $K_{1,m} \diamond C_n$.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest

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