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Connected Size Ramsey Numbers for The Pair Complete Graph of Order Two versus Union Complete Graph of Order Three

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Abstract

Let F, G, and H be finite, simple, and undirected graphs. The connected size Ramsey number $\hat{r}_c(G, H)$ of graph G and H is the least integer k such that there is a connected graph F with k edges and if the edge set of F is arbitrarily colored by red or blue, then there always exists either a red copy of G or a blue copy of H. This paper shows that the connected size Ramsey number $\hat{r}_c(2K_2, nK_3) = 4n + 3$, for $n \geq 4$.

Keywords: graph; connected; size Ramsey number; complete graph.

1. INTRODUCTION

Let F, G, and H be simple and finite graphs. We use the notation $|E(G)|$ and $\Delta(F)$ to denote the size and maximum degree of respectively. Draganić and Petrova, K. recently discussed the size of Ramsey numbers of graphs with maximum degree [2]. Let $v \in V(G)$, the degree of a vertex v, denoted by $d(v)$, is the number of edges incident to the vertex v. The neighborhood $N_G(v)$ of vertex v is the set of vertices adjacent to v in G. For Every red-blue coloring of the edges of graph F, there exists either a red subgraph G or a blue subgraph H in F, we write $F \longrightarrow (G, H)$. A red-blue coloring in F such that neither a red G nor a blue H occurs is called a (G, H)-coloring. A graph G is said to be decomposable into subgraph $H_1, H_2, \dots, H_t, E(H_i) \neq 0$ for every i, if any two subgraphs H_i and H_j have no edges in common, and the union for all the subgraphs H_i is G.

A path graph P with $n \geq 1$ is the graph whose vertices can be listed in the order $u_1, u_2, u_3, \dots, u_n$ such that $E(P) = \{u_i u_{i+1}, i = 1,2, \dots, n-1\}$. The path graph with n vertices is denoted by P_n, while cycle graph of order n is denoted by C_n is graph with the vertex set $V(C_n) = V(P_n)$ and the edge set $E(C_n) = E(P_n) \cup \{u_1 u_n\}$. If every 2 verteices of any graph G is connected by a path, then the graph is called *connected graph.* The graph of order n, in which every pair of distinct vertices is adjacent, is called the *complete graph* and denoted by K_n.

Any coloring of the edges of F, say red-blue, contains a red subgraph G or a blue subgraph H, then we say F *arrowing* G and H, denoted by $F \rightarrow (G, H)$. On the other hand, a red-blue coloring of F called $(G, H) - good$ if F does not contain a red subgraph G and a blue subgraph H in this coloring. Then, $F \nrightarrow (G, H)$ if there is a red-blue coloring of F that $(G, H) - good$. The *size Ramsey number* of a pair graph G and H, denoted by $\hat{r}(G, H)$, is min $\{ |E(F)| : F \rightarrow (G, H)\}$ and the *connected size Ramsey number* of a pair graph G and H, denoted by $\hat{r}_c(G, H)$, is defined as min $\{ |E(F)| : F \rightarrow (G, H), F \text{ is connected}\}$. In this paper, graph F is called a power graph. Erdös at all. [3] in 1978 introduced the size Ramsey number of graph. They studied for a pair of complete graphs, stars, and the products of two graphs [3]. Let nK_2 denote the graph consisting of n independent edges and P_n be a path on n vertices. [3] showed that for $n \geq 1$,

$$\hat{r}(nK_2, P_4) = \left\lceil \frac{5n}{2} \right\rceil$$

and

$$\hat{r}(nK_2, P_5) = \begin{cases} 3n, & \text{if } n \text{ is even,} \\ 3n + 1, & \text{if } n \text{ is odd.} \end{cases}$$

They also showed that for fixed $t \geq 2$, there are positive constants a and b such that for all ≥ 3, $n + a\sqrt{n} < \hat{r}(tK_2, C_n) < n + b\sqrt{n}$. The size Ramsey number $\hat{r}(t2, C_n)$ is $2n$ and the size Ramsey number $\hat{r}(nK_2, P_4) = \begin{cases} 3n - 1, \text{for even } n, \\ 3n, \text{ for.} \end{cases}$

A few years later, Javadi et al. gave the upper bound in general to size Ramsey number of Cycle graph is $\hat{r}(C_n, C_n) = 10^6 \times cn$, $c = 843$ for n is even and $c = 113482$ is odd [7]. Several years later, Li et al. showed the size Ramsey number for the path versus path and path versus cycle [15]. A direct consequence of the concept of the size Ramsey number and Ramsey number in graph is the restricted size Ramsey number [13]. They discussed the restricted size Ramsey number for $2K_2$ versus dense connected graphs of order six.

Rahajeng et al. in 2015 introduced the connected size Ramsey numbers for matchings versus cycles or paths with the result: for $n \geq 4$, $\hat{r}_c(2K_2, C_n) = 2n$, and

$$\hat{r}_c(nK_2, P_4) = \begin{cases} 3n - 1, & \text{if } n = 2,4, \\ 3n, & \text{if } n = 3,5. \end{cases}$$

They also showed that

$$\hat{r}_c(nK_2, P_4) \leq \begin{cases} 3n - 1, & \text{if } n \text{ is even}, \\ 3n, & \text{if } n \text{ is odd}. \end{cases} \quad [11]$$

The other result's Rahajeng et al. in ([9]; [10]) are the size Ramsey number and Ramsey number for matchings versus stars. The upper bounds for $\hat{r}_c(tK_2, P_m)$, $t \geq 1$ given by Vito at al. in [14] and after that Zhang et al. in [16] presented the exact value of the connected size Ramsey numbers for matchings versus paths i.e. $\hat{r}_c(nK_2, P_4) = 3n - 1$ if n is even and otherwise is $\hat{r}_c(nK_2, P_4) = 3n$. A year later, Wang et al. in [12] shown that $\hat{r}_c(nK_2, C_3) = 4n - 1$ for all positive integer n. In 2017, Indrayani et al studied the connected size Ramsey number $\hat{r}_c(2K_2, nK_3)$, for $n = 1,2,3$ and the exact value's are $7, 11,$ and 15 [6]. The other researchers who study about size Ramsey number $\hat{r}_c(2K_2, H)$ can be seen in ([1]; [4]; [5]; [8]). Motivated by the above results, in this paper, we will study all the connected size Ramsey numbers of $\hat{r}_c(2K_2, nK_3)$, for $n \geq 4$.

2. METHODS

2.1. Types, subjects, and research objects

The type of research is a literature study conducted by studying and examining references in the form of four books, five e-books, nine journal articles, and seven proceedings that are relevant to the research topic. The subject of this research is the connected size Ramsey number $\hat{r}_c(G,H)$, for $G = 2K_2$ and H. The objects studied in this study are the graphs resulting from comb operations cycle C_n versus star $S_{1,n}$ and fan $F_{1,n}$.

2.2 Methods in research

First, determine the upper bound, i.e., choose a graph F that satisfies $F \rightarrow (2K_2, kK_3)$. At this stage, we found $\hat{r}_c(2K_2,\ kK_3) \leq |F|$. Next, we determine the lower bound, i.e. take any graph F' that satisfies $F' \nrightarrow (2K_2, kK_3)$. Finally, we found $\hat{r}_c(2K_2,\ kK_3) \geq |F'| + 1$ and obtained $|F| = |F'| + 1$.

3. RESULTS AND DISCUSSION

Let $\hat{r}_c(G,H) = k$ is the connected size Ramsey number for graph G and H, which means the least integer k such that there is a connected graph F with k edges and if the edge set of F is arbitrarily colored by red or blue, then there always exists either a red copy of G or a blue copy of H. In such a case is written $\hat{r}_c(G,H) = k$, which means $\hat{r}_c(G,H) \leq k$ and $\hat{r}_c(G,H) \geq k$. For $\hat{r}_c(G,H) \leq k$, the value of k is called the upper bound for $\hat{r}_c(G,H)$ and for $\hat{r}_c(G,H) \geq k$, the value of k is called the lower bound. Based on the definition of the connected size Ramsey number, there are two processes in this research. The process of determining the upper bound and the process of determining the lower bound. The following is the process of determining the connected size Ramsey number for graph $G = 2K_2$ and $H = nK_3$, for $n \geq 4$ which always starts with the determination of the upper bound.

The following is given one theorem as the main result of this research. A saw graph is used as a power graph to prove the main result. Before presenting the main results of this study, the definition of a saw graph is first given.

Definition 3.1 (*k*-type *Saw graph*). *Let path P_n with $V(P_n) = \{v_1, v_2, v_3, v_4, \dots, v_{n-1}, v_n\}$ and $V\left(\left\lceil \frac{n}{k+1} \right\rceil K_1\right) = \left\{u^1, u^2, \dots, u^{\left\lceil \frac{n}{k+1} \right\rceil}\right\}$, $k \geq 1$.* ***Saw graph of type*** *k denoted by $\boldsymbol{Gr_n^k}$ is a graph with vertex set $V(\boldsymbol{Gr_n^k}) = V(P_n) \cup V\left(\left\lceil \frac{n}{k+1} \right\rceil K_1\right)$ and edge set $\boldsymbol{E}(\boldsymbol{Gr_n^k}) = E(P_n) \cup \left\{u^j v_{j+(j-1)k}, u^j v_{j+(j-1)k+1}\ : j = 1,2, \dots, \left\lceil \frac{n}{k+1} \right\rceil\right\}$.*

Example 3.1 Take $n = 8$ and $k = 1$. Saw graph Gr_8^1 has vertex set $V(Gr_8^1) = \{v_1, v_2, v_3, v_4, \dots, v_7, v_8\} \cup \{u^1, u^2, u^3, u^4\}$ with $V(P_8) = \{v_1, v_2, v_3, v_4, \dots, v_7, v_8\}$ and $V(4K_1) = \{u^1, u^2, u^3, u^4\}$. By Definition 3.1, edge set of Gr_8^1 is $E(Gr_8^1) = \{v_1v_2, v_2v_3, \dots, v_6v_7, v_7v_8\} \cup \{u^1v_1, u^1v_2, u^2v_3, u^2v_4, u^3v_5, u^3v_6, u^4v_7, u^4v_8\}$. The saw graph structure Gr_8^1 is shown in Figure 3.1 contains $\left\lceil \frac{n}{2} \right\rceil = 4$ independent cycles.

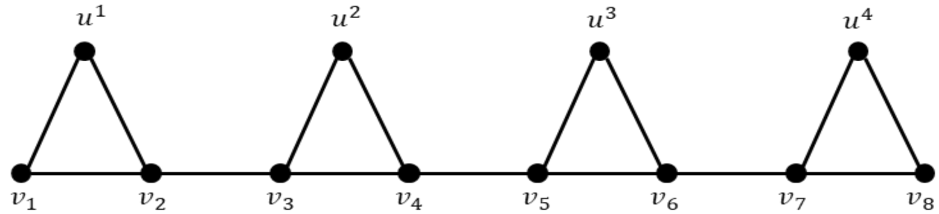

Figure 3.1. Saw graph Gr_8^1.

Theorem 3.1. $\hat{r}_c(2K_2, 4K_3) = 19$.

Proof. The connected size Ramsey number for $2K_2$ versus $4K_3$ is the least integer k such that if the edge set of F with $|E(F)| = k$ is arbitrarily colored by red or blue, then there always exists either a red copy of G or a blue copy of H. Can be written:

$$\hat{r}_c(2K_2, 4K_3) = \min\{|E(F)| : F \rightarrow (2K_2, 4K_3), F \text{ is connected}\}.$$

So in this proof, we will show that $\hat{r}_c(2K_2, 4K_3) = 19$ by showing $F \rightarrow (2K_2, 4K_3)$ for a given connected graph F and $|E(F)|$ is minimum. Select the saw graph $F = Gr_{10}^1$ as shown in Figure 3.2. Let $V(Gr_{10}^1) = \{v_j : 1 \le j \le 10\} \cup \{u^i : 1 \le i \le 5\}$ and $E(Gr_{10}^1) = \{v_i v_{i+1} : 1 \le i \le 9\} \cup \{u^j v_{j+(j-1)}, u^j v_{j+(j-1)+1} : j = 1,2,\dots,5\}$. We can see that $|E(F)| = 19$.

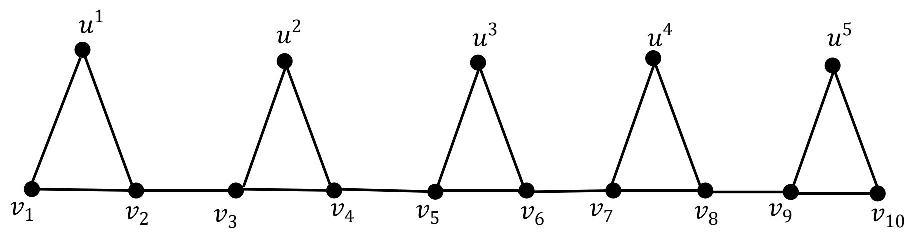

Figure 3.2. The Saw Graph Gr_{10}^1.

Suppose we color red and blue on all edges of Gr_{10}^1, as shown in Figure 3.3.

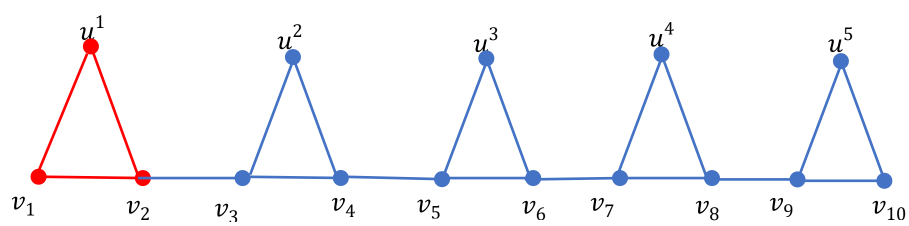

Figure 3.3. The Color Saw Graph Gr_{10}^1

The color saw graph Gr_{10}^1 does not contain red $2K_2$ but contains blue $4K_3$. In general, assume the saw graph Gr_{10}^1 does not contain the red $2K_2$, namely: $u^j v_{j+(j-1)+1}, v_{j+(j-1)} v_{j+(j-1)+1}$ and $v_{j+(j-1)+1} v_{j+(j-1)+2}$, then the induced subgraph $F[V(Gr_{10}^1) \backslash \{v_{j+(j-1)+1}\}]$ contains the blue $4K_3$ like in Figure 3.4.

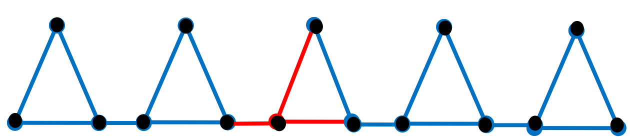

Figure 3.4. The red edges in Gr_{10}^1.

Therefore, any red and blue coloring on all edges of graph Gr_{10}^{1} always contains the red $2K_2$ or the blue $4K_3$. So we have,

$$\hat{r}_c(2K_2,\ 4K_3) \leq 19. \qquad \ldots(1)$$

Next, we show that $\left|E(Gr_{10}^{1})\right| = 19$ is the minimum value. Let the connected graph F' with $|E(F')| = 18$. The structure of graph F' is various, and it will be divided into two cases.

Case 1: Graph F' is a connected graph of size 18, which does not contain K_3 as a subgraph.

All edges on graph F' are colored by blue. So, graph F' contains no red $2K_2$ as a subgraph and no blue $4K_3$ as a subgraph (Figure 3.5).

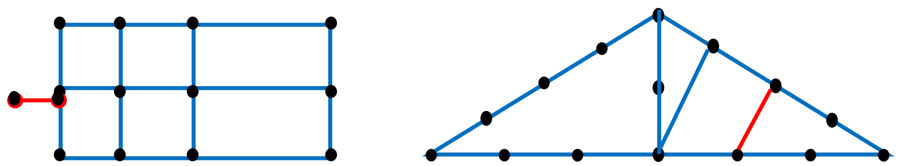

Figure 3.5. The graph GR_{10} contains no K_3.

Case 2: Let F' be the connected graph of size 18 that contains the K_3 subgraph.

If graph F' contains K_3 as a subgraph, then there is at least one vertex, namely, u in K_3 is at least 3 degrees or to one vertex is 4 degrees because of the incident in $2K_3$. Paint all the edges incident with the u vertex in red on F'. The other edges are colored in blue. Therefore, there are at least 3 edges of the red color on the graph F'. Two edges of the three edges of red are included in K_3. That means one edge is blue but disjoints the other blue edges. Consequently, there is a K_3 that does not form blue K_3. Because graph F' can contain subgraphs at most $4K_3$ and two or three edges adjacent to graph F' are colored red (no red $2K_2$), thus the color graph F' at most contains the blue $3K_3$ as a subgraph (Figure 3.6).

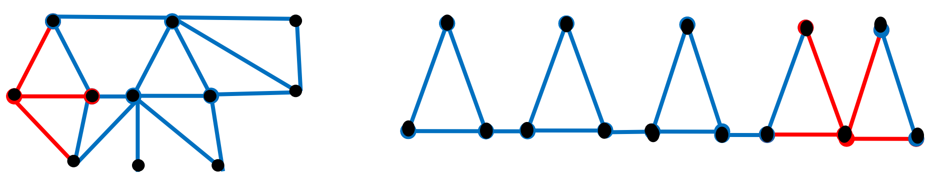

Figure 3.6. Red coloring on any graph F' of size 18.

From uppercase 1 and Case 2, it is known that in any connected graph F' of order 18, there is always coloring on the graph which does not contain the red $2K_2$ and also does not contain the blue $4K_3$, denoted $F' \nrightarrow (2K_2, 4K_3)$. Thus, we obtain $\hat{r}_c(2K_2,\ 4K_3) > 18$ or can be written:

$$\hat{r}_c(2K_2, 4K_3) \geq 19. \qquad \ldots (2)$$

Based on inequalities (1) and (2), it is known that $Gr_{10}^{1} \rightarrow (2K_2, 4K_3)$, and $\left|E(Gr_{10}^{1})\right| = 19$ is minimum, such that $\hat{r}_c(2K_2, 4K_3) = 19$. ■

In a similar way to Theorem 3.1, we get $\hat{r}_c(2K_2,\ 5K_3) = 23$.

Theorem 3.2. *For* $k \in \mathbb{N}$ $\hat{r}_c(2K_2,\ kK_3) = 4\ (k) + 3$.

Proof. First, we prove that $\hat{r}_c(2K_2,\ kK_3) \leq 4\ (k) + 3$. Select the saw graph $F = Gr_{2(k+1)}^{1}$ with $V\left(Gr_{2(k+1)}^{1}\right) = \left\{v_j\text{: } 1 \leq j \leq 2(k+1)\right\} \cup \left\{u^i\text{: } 1 \leq i \leq k+1\right\}$ and $E\left(Gr_{2(k+1)}^{1}\right) = \{v_i v_{i+1}\text{: } 1 \leq i \leq 2k+1\ \} \cup \left\{u^j v_{j+(j-1)}, u^j v_{j+(j-1)+1}\ : j = 1,2,\ldots,\mathrm{k}+1\right\}$. Assume the saw graph $Gr_{2(k+1)}^{1}$ does not contains the red $2K_2$, namely: $u^j v_{j+(j-1)+1}, v_{j+(j-1)} v_{j+(j-1)+1}$ and $v_{j+(j-1)+1} v_{j+(j-1)+2}$ for a $j \in \{1,2,\dots,\mathrm{k}+1\}$ are the red edges, then the color connected graph $Gr^1_{2(k+1)}$ contains no the red $2K_2$. The graph $Gr^1_{2(k+1)}$ has$\left|E(Gr^1_{2(k+1)})\right| = 4\,(k) + 3$ and $(k+1)K_3$ and all edges on $E\left(Gr^1_{2(k+1)}\right)\backslash\{u^j v_{j+(j-1)+1}, v_{j+(j-1)} v_{j+(j-1)+1},\ v_{j+(j-1)+1} v_{j+(j-1)+2}\}$ have blue color. Because the red edges: $u^j v_{j+(j-1)+1}, v_{j+(j-1)} v_{j+(j-1)+1}$ and $v_{j+(j-1)+1} v_{j+(j-1)+2}$ lie in a cycle K_3, then the color graph $Gr^1_{2(k+1)}$ contains the blue kK_3 as a subgraph. Hence, any red and blue coloring on all edges of graph $Gr^1_{2(k+1)}$ always contains the red $2K_2$ or the blue kK_3. It means $Gr^1_{2(k+1)} \rightarrow (2K_2, kK_3)$. So we have,

$$\hat{r}_c(2K_2,\ kK_3) \leq 4k + 3. \qquad \dots(3)$$

Next, we will show that $\hat{r}_c(2K_2,\ kK_3) \geq 4k + 3$ by showing that any graph F with $|E(F)| = 4k + 2$, graph $F \nrightarrow (2K_2, kK_3)$. We know that the structure of the graph F with $|E(F)| = 4k + 2$ is various, and here, it will be divided into two cases.

Case 1: Graph F is a connected graph of size $4k + 2$, which does not contain K_3 as a subgraph.

All edges on graph F are colored by blue. So, graph F contains no red $2K_2$ as a subgraph and no blue kK_3 as a subgraph. In this case

Case 2: Let F be the connected graph of size $4k + 2$ that contains the K_3 subgraph.

If graph F contains K_3 as a subgraph, then there is at least one vertex, namely, u in K_3 is at least 3 degrees or one vertex is 4 degrees because incident in $2K_3$. Paint all the edges incident vertex u in red on F and the other edges are blue color. Therefore, there are at least 3 edges of the red color on graph F. There are at least two edges of the three edges of red included in K_3. That means, at most, one edge is blue but disjoint with other blue edges. Consequently, there is a K_3 that does not form blue K_3. Because graph F can contain subgraphs at most kK_3 and two or three edges are adjacent in graph F are colored by red (no red $2K_2$), such that the color graph F at most contains the blue $(k-1)K_3$ as a subgraph.

Case 1 and Case 2 are known that in any connected graph F of order $4k + 2$, there is always coloring on the graph which does not contain the red $2K_2$ and also does not contain the blue kK_3, denoted $F \nrightarrow (2K_2, 4K_3)$. So,

$$\hat{r}_c(2K_2, kK_3) \geq 4k + 3. \qquad \dots(4)$$

By (3) and (4), we have $\hat{r}_c(2K_2, kK_3) = 4k + 3$. ■

4. CONCLUSION

In determining the connected size Ramsey number for the pair complete graph of order two versus the union complete graph of order three, we conclude that the upper bound value is equal to the lower bound value. Therefore, the connected size Ramsey number for the pair complete graph of order two versus the union complete graph of order three exactly is $\hat{r}_c(2K_2, nK_3) = 4n + 3$, for $n \geq 4$.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this article. The authors confirmed that the paper was free of plagiarism.

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