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Hopf bifurcation in a dynamic mathematical model in facultative waste stabilization pond

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Abstract

In this paper, we discuss the predator-prey model using Holling type II functional response with the time delay in facultative stabilization pond. In this research, we discuss the predator-prey model using Holling type II functional response with the time delay, determining the equilibrium point, the stability analysis of predator-prey model using Holling type II functional response with the time delay and numerical simulation of the predator-prey model using Holling type II functional response with the time delay. The method used to analyse the problem is by literature study. The steps used are the development of a mathematical model of change of dissolved oxygen concentration, phytoplankton and zooplankton, mathematical equation solving algorithm, field data, simulation using Maple and Mathematica 9 software and validation with research. As a result of the research, from the model obtained there are 3 equilibrium points, i.e. $T_{0,}$ T_1 and T_2 with conditions $2\alpha\beta + \varphi(\varphi - 1) > 0, \alpha\beta > \varphi, \omega > \mu$ and $\frac{\beta C}{C+\varphi} > \frac{\mu h}{\omega - \mu}$. To analyze the existence of Hopf bifurcation, the predator-prey population dynamics was simulated based on three cases, by decreasing the time-delay in the growth rate of the predator population (τ_k). By chossing an exact parameter value (τ_k), we can showed the existence of Hopf bifurcation. In the case $\tau = \tau_k$ the stable spiral changed into an unstable spiral and also observed the presence of limit cycles. This is known as Hopf bifurcation. Then, to illustrate the model, simulation model. The model simulations give the same result with the analysis.

Keywords: Hopf Bifurcation, dynamic mathematical model, stabilization pond

Abstrak

Pada tulisan ini dibahas model predator-prey menggunakan fungsi respon Holling tipe II dengan waktu tunda pada kolam stabilisasi fakultatif. Pada penelitian ini dibahas model predator-prey menggunakan fungsi respon Holling tipe II dengan waktu tunda, penentuan titik kesetimbangan, analisis kestabilan model predator-prey menggunakan fungsi respon Holling tipe II dengan waktu tunda dan simulasi numerik dari model predator-prey menggunakan fungsi respon Holling tipe II dengan waktu tunda. Metode yang digunakan untuk menganalisis permasalahan adalah dengan studi literatur. Langkah-langkah yang digunakan adalah pengembangan model matematis perubahan konsentrasi oksigen terlarut, fitoplankton dan zooplankton, algoritma penyelesaian persamaan matematis, data lapangan, simulasi menggunakan software Maple dan Mathematica 9 dan validasi dengan penelitian. Sebagai hasil penelitian, dari model yang diperoleh terdapat 3 titik kesetimbangan, yaitu $T_{0,}$ T_1 dan T_2 dengan syarat $2\alpha\beta + \varphi(\varphi - 1) > 0, \alpha\beta > \varphi, \omega > \mu$ dan $\frac{\beta C}{C+\varphi} > \frac{\mu h}{\omega - \mu}$. Untuk menganalisis keberadaan bifurkasi *Hopf*, dinamika populasi *predator-prey* dibagi menjadi tiga kasus dimana tiap kasusnya mengalami penurunan nilai parameter tingkat

pertumbuhan populasi *predator*. Dengan memilih nilai parameter yang tepat (τ_k), dapat ditunjukkan keberadaan dari bifurkasi *Hopf*. Pada kasus $\tau = \tau_k$ terjadi perubahan kestabilan T_2 dari spiral stabil *menjadi* tak stabil. Fenomena ini merupakan sifat dari bifurkasi *Hopf*. Selanjutnya, untuk mengilustrasikan model tersebut maka dilakukan simulasi. Simulasi model yang dilakukan memberikan hasil yang sama dengan hasil analisis.
Kata Kunci: Bifurkasi Hopf, model matematika dinamis, kolam stabilisasi

1. INTRODUCTION

The research of the interaction of prey-predator will be done analyzing the mathematical model. Our model based on the basic phytoplankton-oxygen model of Sekerci and Petrovskii research[11] which we expanded taking into account zooplankton and considering the influence of predation on oxygen dynamics. The functional response in ecology is the amount of food eaten by a predator as a function of food density. In this case the functional response is divided into three types, i.e. type I, II and III functional response. Type II functional response occurs in predators whose characteristics are active in searching for prey.

To build a more realistic model, consider the delay time. Time delay is important in modeling real problems, because decisions are usually made based on previous conditions. This is important in modeling population decline, because the rate of population decline not only depends on the population size at one time, but also depends on previous times.

This model will be applied to the wastewater treatment system at the Sewon Wastewater Treatment Plant (IPAL) using DO (Dissolved Oxygen) benchmarks and biological species indicators (phytoplankton and zooplankton) to determine the quality of processing.

There is a large body of literature reviewing various aspects of spatiotemporal (in space and time) plankton dynamics. The predator-prey conceptual model for describing phytoplankton and zooplankton interactions in aquatic ecosystems was considered in detail by Malchow et al but without considering oxygen production. In another mathematical study, Edwards and Brindley examined the dynamics of combined plankton-nutrient systems, but did not consider possible plankton-nutrient relationships with dissolved oxygen. There are few studies in which oxygen production is explicitly considered. Specifically, Marchettini et al studied trophic dynamics by developing mathematical models of biochemical processes in lagoon ecosystems. The dissolved oxygen concentration in a multi-component system is considered by them. In another modeling study, Allegretto et al demonstrated the existence of periodic solutions in modeling Italian coastal lagoons.

Marchettini et al studied a mathematical model of the biochemical processes of the lagoon system. In another study, Allegretto et al focused on the presence of periodic fluctuations, based on Italian coastal lagoons. Dynamics of plankton-nutrient systems and their dynamic properties. Additionally, an oxygen-algae model is introduced to reduce oxygen depletion under some external controlling factors. Hull et al.[4] investigated seasonal and daily dynamics of dissolved oxygen measurements in Mediterranean coastal lagoons. Another plankton-oxygen model has been proposed and analyzed by Misra [7] including the effect of some ‘exogenous’ factors (such as light, wind intensity, temperature, phosphorus, eutrophication, etc.), hence leaving the internal plankton-oxygen dynamics out of the focus. Misra [8] propose and analyze a non-linear mathematical model for algal bloom in a lake to account for the delay in conversion of detritus into nutrients. It is assumed that there is a continuous inflow of nutrients in the lake due to agricultural run off.

In this research, we will discuss the modification of the predator-prey model using Holling type II with time delay, determining the equilibrium point, stability analysis and numerical simulation of the predator-prey model using Holling type II with time delay. Holling type II was chosen because it has problems that are in accordance with this type of predator which is characterized by being active in looking for prey.

2. PRELIMINARIES

The nonlinear differential equation is given as following.

$$\dot{x} = f(x). \qquad (1)$$

The Eq. (1) has an equilibrium point $x = \bar{x}$ if it satisfies $f(\bar{x}) = 0$.
For the nonlinear differential equation, the stability analysis uses the linearization. Suppose that we linearize the Eq. (1), then can be written as

$$\dot{x} = Ax + \varphi(x). \qquad (2)$$

It is known from [2] that the Eq. (2) is the nonlinear differential equation with A the Jacobian matrix of the Eq. (1) at the equilibrium point $\bar{x}$,

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\bar{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\bar{x}) \end{pmatrix} \qquad (3)$$

and $\varphi(x)$ as the linear part of the Eq. (1). Ax in the Eq. (2) is called the linearization of the nonlinear Eq. (1) which is written in the form $\dot{x} = Ax$.

Suppose that A matrix $n \times n$ and $x \in \mathbb{R}^n, x \neq 0$. The vector x is called the eigenvector or the characteristic vector of A if

$$Ax = \lambda x, (4)$$

for a $\lambda \epsilon \mathbb{R}$, the number λ that satisfy the above equation is called the eigenvalue or the characteristic value. To find the eigenvalue of the matrix $n \times n$ then the Eq. (4) can be rewritten as

$$(A - \lambda I)x = 0, \qquad (5)$$

with I is the identity matrix. The Eq. (5) has the nontrivial solution if

$$det(A - \lambda I) = |A - \lambda I| = 0. \qquad (6)$$

The Eq. (6) is called the characteristic equation of the matrix A [3]. Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. From the Eq. (6), then the characteristic equation of A becomes $\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$. Therefore, we obtain the equation $\lambda^2 - \tau\lambda + \Delta = 0$, with $\tau = \text{trace}\,(A) = a + d = \lambda_1 + \lambda_2$ and $\Delta = \det(A) = ad - bc = \lambda_1\lambda_2$. Such that, we have the eigenvalues from the matrix A as following.

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}. \qquad (7)$$

According to [1], to determine the stability of the fixed point of system can be shown from the value Δ. There are three cases for the value Δ i.e.:

(1) $\Delta < 0$

If two eigenvalues have the different sign, then the fixed point is saddle.

(2) $\Delta > 0$

(a) $\tau^2 - 4\Delta > 0$

(i) If $\tau > 0$ and both eigenvalues are the positive real number, then the fixed point is unstable node.

(ii) If $\tau < 0$ and both eigenvalues are the negative real number, then the fixed point is stable node.

(b) $\tau^2 - 4\Delta < 0$

(i) If $\tau > 0$ and both eigenvalues are the complex number, then the fixed point is unstable spiral.

(ii) If $\tau = 0$ and both eigenvalues are the complex number, then the fixed point is stable spiral.

(iii) If $\tau < 0$ and both eigenvalues are the complex number, then the fixed point is center.

(c) $\tau^2 - 4\Delta = 0$

Parabola is the boundary line between the node and spiral. Star node or degenerate lies in this parabola. If both eigenvalues have the same value, then the fixed point is proper node.

(3) $\Delta = 0$

If one of the eigenvalue is zero, then the fixed point is not the isolated equilibrium point.

Namely, given a real polynomial

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \qquad (8)$$

with positive and real numbers, $k = 1, 2, 3, \cdots n$

The $n \times n$ square matrix

$$H = \begin{bmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix} \qquad (9)$$

is called Hurwitz matrix corresponding to the polynomial p. It was established by Adolf Hurwitz in 1895 that a real polynomial is stable (that is, all its roots have strictly negative real part) if and only if all the leading principal minors of the matrix $H(p)$ are positive:

$$\Delta_1(p) = |a_1| = a_1 > 0, \Delta_2(p) = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_1 a_2 - a_0 a_3 > 0, \Delta_n(p) = |H| > 0. [10] \qquad (10)$$

Furthermore, Guckenheimer and Holms in Irwan [6] explain that the qualitative structure of a dynamic system can change due to the change of the parameter of the dynamic system. This is known a bifurcation. Bifurcation is the change in the stability and the number of the equilibrium point due to the change in the parameter. Suppose that the equation of the differential equation system

$$\begin{cases} \dot{x} = f(x, y, \mu) \\ \dot{y} = g(x, y, \mu) \end{cases}. \qquad (11)$$

We assume that Eq. (11) has the equilibrium point (x^*, y^*) and $\mu = \mu^*$ is the value of the parameter that caused the bifurcation.

According to [5], the Hopf bifurcation occurs if the equilibrium point (x^*, y^*) has a pair of the complex eigenvalues i.e. $\lambda(\mu^*) = p(\mu^*) + iq(\mu^*)$ with $p(\mu^*) = 0, q(\mu^*) \neq 0$ and satisfies the transversal condition.

3. METHOD

The stages of the research carried out are as follows. Journal review, development of a mathematical model of change of dissolved oxygen concentration, phytoplankton and zooplankton, the stage of determining the fixed point is obtained by making the rate of change of predator and prey with respect to time equal to zero, fixed point stability analysis stage where fixed point stability without time delay is obtained by a linear approach whereas fixed point with a time delay requires an approach in complex space to analyze the Hopf bifurcation, Stage of determination of transverse conditions to prove that a Hopf bifurcation occurs at the inner equilibrium point and the interior point stability simulation stage is carried out for each parameter according to the conditions so that obtained an overview of the influence of delay time on predators and prey.

4. MODEL FORMULATION

4.1.THE BASELINE MODEL

We begin with a simple conceptual model that only takes into account the temporal dynamics of the oxygen itself and the phytoplankton as its main producer:

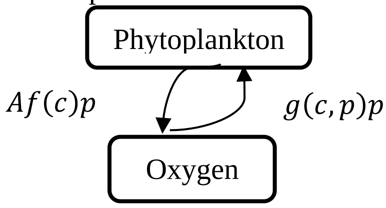


Figure 1. Interactions between oxygen & phytoplankton. Phytoplankton produces oxygen through photosynthesis during the day-time depending on existence of sunlight and consumes it during the night

$$\frac{dc}{dt} = Af(c)p - mc, \qquad (12)$$

$$\frac{dp}{dt} = g(c,p)p, \qquad (13)$$

here c and p the concentration of the dissolved oxygen and the phytoplankton density $f(c)$: the amount of oxygen produced per unit time and per unit phytoplankton mass, $g(c,p)$: the per capita phytoplankton growth rate, A: a coefficient that can take into account the effect of relevant environmental factors and mc: oxygen losses, e.g. due to its diffusion to the atmosphere, plankton breathing, etc. Note that Eq. (12) is linear with respect to p and indeed we are not aware about any evidence that the photosynthesis rate can depend on phytoplankton density. On the contrary, Eq. (13) should normally be nonlinear with respect to p (hence the dependence of g on p) as the high phytoplankton density is known to damp its growth, e.g. due to self shading and or nutrient depletion. In order to understand what can be the properties of functions f and g, we have to look more closely at the oxygen production and consumption. Consider $f(c)$ first. Oxygen is produced inside phytoplankton cells in photosynthesis and then diffuse through the cell membrane into the surrounding water. Diffusion flux always directed from areas with higher concentration of the diffusing substance to the areas with lower ones; the larger is the difference between the concentrations, the larger is the flux (cf. the Fick law). Therefore, for the same rate of photosynthesis, the amount of oxygen that gets through the cell membrane will be the larger the lower is the oxygen concentration in the surrounding water. Therefore, f should be a monotonously decreasing function of c. We further assume that the oxygen flux through the cell membrane tends to zero when the oxygen concentration in the water is very large, i.e., in physical terms, is close to its saturating value $c \to \infty$. The above features are qualitatively taken into account by the following parameterization:

$$f(c) = 1 - \frac{c}{c+c_0}, \qquad (14)$$

where c_0: the half-saturation constant. Considering phytoplankton multiplication, we assume that $g(c,p) = \alpha(c) - \gamma p$ where $\alpha(c)$: the phytoplankton linear growth and γp: intraspecific competition for resources. Eq. (13) for the phytoplankton growth is therefore essentially the logistic growth equation where $\frac{1}{\gamma}$: plays the role of the carrying capacity, which we assume does not depend on c. However, the linear growth rate α should depends on c, which can be seen from the following argument. Phytoplankton produce oxygen in photosynthesis during the daytime, but it needs oxygen for breathing during the night; therefore, a low oxygen concentration is unfavorable for phytoplankton and is likely to depress its reproduction. On the other hand, a phytoplankton cell cannot take more oxygen than it needs. Hence α should be monotonously increasing function of c tending to a constant value for $c \to \infty$. The simplest parameterization for α is then given by the Monod function, so that for $g(c,p)$ we obtain:

$$g(c,p) = \frac{Bc}{c+c_1} - \gamma p, \qquad (15)$$

where c_1: the half-saturation constant and B: the phytoplankton maximum per capita growth rate. With Eqs. (14–15), Eqs. (12–13) take the following form.

$$\frac{dc}{dt} = A\left(1 - \frac{c}{c+c_0}\right)p - mc, \quad (16)$$
$$\frac{dp}{dt} = \left(\frac{Bc}{c+c_1} - \gamma p\right)p. \quad (17)$$
$$t' = tm, c' = \frac{c}{c_0}, p' = \frac{\gamma p}{m}, \hat{A} = \frac{A}{c_0\gamma}, \hat{B} = \frac{B}{m}, \hat{c_1} = \frac{c_1}{c_0}.$$

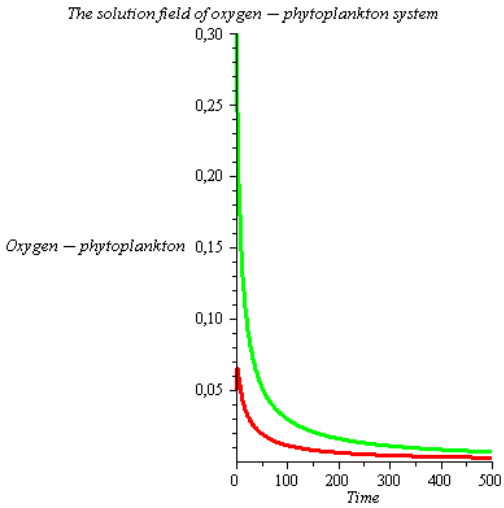
In the equation system (16-17), the parameter $t' = tm \Leftrightarrow t = \frac{t'}{m}$ describes that multiple of time t, $c' = \frac{c}{c_0} \Leftrightarrow c = c'c_0$ shows that oxygen concentration dissolved accompanied by the influence of water saturation level, $p' = \frac{\gamma p}{m} \Leftrightarrow p = \frac{mp'}{\gamma}$ shows that phytoplankton population density, $\hat{A} = \frac{A}{c_0\gamma} \Leftrightarrow A = \hat{A}c_0\gamma$ describes that the influence of environmental factors on the rate of oxygen production which depends on the level of water saturation, $\hat{B} = \frac{B}{m} \Leftrightarrow B = \hat{B}m$ shows that the maximum growth rate of phytoplankton per capita and $\hat{c_1} = \frac{c_1}{c_0} \Leftrightarrow c_1 = \hat{c_1}c_0$ maximum growth saturation level of phytoplankton.

Thus, Eqs. (16-17) is equivalent to the following equation.

$$\frac{dc}{dt} = A\left(1 - \frac{c}{c+c_0}\right)p - mc, \quad (18)$$
$$\frac{dp}{dt} = \left(\frac{Bc}{c+c_1} - p\right)p. \quad (19)$$

Eqs. (16-17) have two equilibrium points, i.e. $T_0(0,0)$ and $T_1\left(\frac{-c_1-1+\sqrt{\Omega}}{2}, \frac{2AB+c_1(c_1-1)-c_1\sqrt{\Omega}}{2A}\right)$ where $\Omega = 4\mathrm{AB} + (c_1-1)^2$ with the conditions $2\mathrm{AB} + c_1(c_1-1) > 0$ and $\mathrm{AB} > c_1$ with Jacobian matrix $J = \begin{pmatrix} -\frac{Ap}{(c+1)^2} - 1 & \frac{A}{c+1} \\ \frac{Bc_1p}{(c+c_1)^2} & \frac{Bc}{c+c_1} - 2p \end{pmatrix}$. From the Jacobian matrix, Eqs. (16-17) calculated at each equilibrium point obtaining the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$ for T_0, $\lambda_{1,2} = \frac{-\rho\pm\sqrt{\rho^2-4\Omega}}{2}$ where $\rho = \frac{A\tilde{p}}{(\tilde{c}+1)^2} - \frac{B\tilde{c}}{\tilde{c}+c_1}$ $+2\tilde{p} + 1$ and $\Omega = \frac{\tilde{p}\tilde{c}(2\tilde{p}\tilde{c}A+4\tilde{p}Ac_1+2\tilde{c}^3+4\tilde{c}^2+2\tilde{c}+4\tilde{c}^2c_1+8\tilde{c}c_1+4c_1+2\tilde{c}{c_1}^2+4{c_1}^2)}{(\tilde{c}+1)^2(\tilde{c}+c_1)^2}$ $+\frac{B\tilde{c}(-\tilde{c}^3-2\tilde{c}^2-\tilde{c}-\tilde{c}^2c_1-2\tilde{c}c_1-c_1+\tilde{p}\tilde{c}A)+\tilde{p}c_1(2\tilde{p}c_1A-AB+2c_1)}{(\tilde{c}+1)^2(\tilde{c}+c_1)^2}$ for $T_1\left(\frac{-c_1-1+\sqrt{\Omega}}{2}, \frac{2AB+c_1(c_1-1)-c_1\sqrt{\Omega}}{2A}\right)$ where $\Omega = 4AB + (c_1-1)^2$.

The simulation results can be shown in Figure 2.



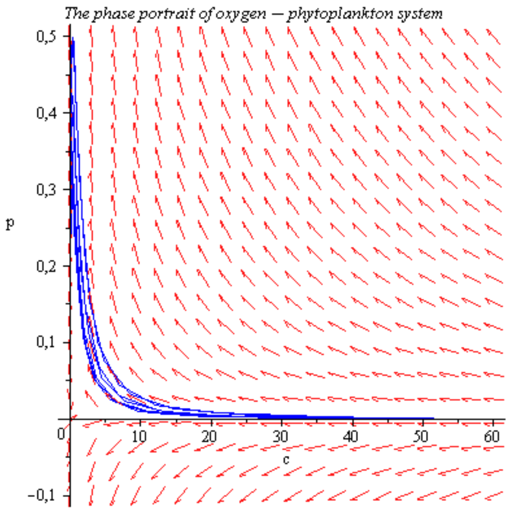


Figure 2. The solution field and phase portrait of the oxygen-phytoplankton system at the equilibrium point $\boldsymbol{T_0}(\mathbf{0},\mathbf{0})$and $\boldsymbol{T_1}\left(\frac{-c_1-1+\sqrt{\Omega}}{2},\frac{2AB+c_1(c_1-1)-c_1\sqrt{\Omega}}{2A}\right)$ where $\boldsymbol{\Omega}=\mathbf{4}\boldsymbol{AB}+(\boldsymbol{c_1}-\mathbf{1})^{\mathbf{2}}$ with $\mathbf{2}\boldsymbol{AB}+\boldsymbol{c_1}(\boldsymbol{c_1}-\mathbf{1})>\mathbf{0}$ and $\boldsymbol{AB}>\boldsymbol{c_1}$

4.2.THE ‘ADVANCE THREE-COMPONENT MODEL

The corresponding model is described by the following differential equations:

$$\frac{dc}{dt}=Af(c)p-c, \quad (20)$$

$$\frac{dp}{dt}=g(c,p)p-e(p,z), \qquad (21)$$

$$\frac{dz}{dt}=e(p,z)-\mu z, \quad (22)$$

where all notations are the same as in section (12-13). Additionally, here z: the zooplankton density at time t, and the function of $e(p,z)$: the per capita zooplankton growth rate due to predation where μ: the zooplankton mortality rate. In the model above, we assume that the phytoplankton-zooplankton interaction is described by the standard prey-predator model with functional response of Holling type II. The second negative term of Eq. (21) corresponds to the grazing of zooplankton on phytoplankton, hence this predation contributes to predator (zooplankton) growth term $\beta e(p,z)$. We consider a Holling type II predator response and use the following parametrization for predation:

$$e(p,z)=\frac{\beta pz}{p+h} \qquad (23)$$

where h: the half-saturation constant and β (dimensionless): maximum per capita growth rate of zooplankton. With Eq. (23) and Eqs. (18-19), then Eqs. (20–22) take the following form:

$$\frac{dc}{dt}=A\left(1-\frac{c}{c+1}\right)p-c, \qquad (24)$$

$$\frac{dp}{dt}=\left(\frac{Bc}{c+c_1}-p\right)\gamma p-\frac{\beta pz}{p+h}, \quad (25)$$

$$\frac{dz}{dt}=\frac{\beta pz}{p+h}-\mu z. \qquad (26)$$

$$t'=tm, c'=\frac{c}{c_0}, p'=\frac{\gamma p}{m}, z'=\frac{\beta z}{m}, \hat{A}=\frac{A}{c_0\gamma}, \hat{B}=\frac{B}{m}, \widehat{c_1}=\frac{c_1}{c_0}, h'=\frac{\gamma h}{m}, \hat{\mu}=\frac{\mu}{m}.$$

In the equation system (26), the parameter $z'=\frac{\beta z}{m}\Leftrightarrow z=\frac{mz'}{\beta}$ describes that zooplankton population density, $h'=\frac{\gamma h}{m}\Leftrightarrow h=\frac{h'm}{\gamma}$ shows that the maximum growth saturation rate of zooplankton and $\hat{\mu}=\frac{\mu}{m}\Leftrightarrow \mu=\hat{\mu}m$ describes that natural death rate of zooplankton.

Thus, Eqs. (24-26) is equivalent to the following equation.

$$\frac{dc}{dt}=A\left(1-\frac{c}{c+1}\right)p-c, \qquad (27)$$

$$\frac{dp}{dt}=\left(\frac{Bc}{c+c_1}-p\right)p-\frac{pz}{p+h}, \qquad (28)$$

$$\frac{dz}{dt}=\frac{\beta pz}{p+h}-\mu z. \qquad (29)$$

Then by giving the discrete time-delay in the rate of zooplankton decline due to predation, the equation models become

$$\frac{dc(t)}{dt}=A\left(1-\frac{c(t)}{c(t)+1}\right)p(t)-c(t), \qquad (30)$$

$$\frac{dp(t)}{dt}=\left(\frac{Bc(t)}{c(t)+c_1}-p(t)\right)p(t)-\frac{p(t-\tau)z(t)}{p(t-\tau)+h}, \qquad (31)$$

$$\frac{dz(t)}{dt}=\frac{\beta p(t)z(t)}{p(t)+h}-\mu z(t). \qquad (32)$$

5. MAIN RESULT

5.1EQUILIBRIA

Theorem 1.

From the above Eqs. (30-32), we obtain:

1. Without the condition, Eqs. (30-32) only has one equilibrium points i.e. the equilibrium point T_0.
2. If $2AB + c_1(c_1 - 1) > 0$ and $AB > c_1$ then Eqs. (30-32) only has two the equilibrium points i.e. the equilibrium point T_0 and T_1.
3. If $\beta > \mu$ and $\frac{B\left(-1+\sqrt{1+\frac{4A\mu h}{\beta-\mu}}\right)}{-1+\sqrt{1+\frac{4A\mu h}{\beta-\mu}}+2c_1} > \frac{\mu h}{\beta-\mu}$ then Eqs. (30-32) only has two equilibrium points i.e. the equilibrium point T_0 and T_2.
4. If $2AB + c_1(c_1 - 1) > 0, AB > c_1, \beta > \mu$ and $\frac{B\left(-1+\sqrt{1+\frac{4A\mu h}{\beta-\mu}}\right)}{-1+\sqrt{1+\frac{4A\mu h}{\beta-\mu}}+2c_1} > \frac{\mu h}{\beta-\mu}$ then Eqs. (30-32) only has two equilibrium points i.e. the equilibrium point T_0, T_1 and T_2.

Proof:

Eqs. (27-29) realizes the equilibrium point when

$$A\left(1 - \frac{c}{c+1}\right)p - c = 0, \qquad (33)$$

$$\left(\frac{Bc}{c+c_1} - p\right)p - \frac{pz}{p+h} = 0, \qquad (34)$$

$$\frac{\beta pz}{p+h} - \mu z = 0. \qquad (35)$$

From Eq. (35) $\frac{\beta\tilde{p}\tilde{z}}{\tilde{p}+h} - \mu\tilde{z} = 0 \Rightarrow \tilde{z} = 0 \vee \tilde{p} = \frac{\mu h}{\beta-\mu}$.

(1) Case $\tilde{z} = 0$. $\left(\frac{B\tilde{c}}{\tilde{c}+c_1} - \tilde{p}\right)\tilde{p} - \frac{\tilde{p}\tilde{z}}{\tilde{p}+h} = 0 \Rightarrow \tilde{p} = 0 \vee \tilde{c} = \frac{\tilde{p}c_1}{B-\tilde{p}}$.

(a) Case $\tilde{z} = 0$ and $\tilde{p} = 0$. $A\left(1 - \frac{\tilde{c}}{\tilde{c}+1}\right)\tilde{p} - \tilde{c} = 0 \Leftrightarrow \tilde{c} = 0$. So we obtain $T_0(0,0,0)$.

(b) Case $\tilde{z} = 0$and $\tilde{c} = \frac{\tilde{p}c_1}{B-\tilde{p}}$. $A\left(1 - \frac{\tilde{c}}{\tilde{c}+1}\right)\tilde{p} - \tilde{c} = 0 \Leftrightarrow \tilde{p} = \frac{\tilde{c}}{A}(\tilde{c}+1)$. Substituting $\tilde{c} = \frac{\tilde{p}c_1}{B-\tilde{p}}$ into the equation $\tilde{p} = \frac{\tilde{c}}{A}(\tilde{c}+1)$ we get $\tilde{p}_{1,2} = \frac{-\sigma\pm\sqrt{\sigma^2-4A\kappa}}{2A}$, where $\sigma = -2AB - {c_1}^2 + c_1$ and $\kappa = AB^2$
$-Bc_1$. Considering $\tilde{p} = \frac{-\sigma-\sqrt{\sigma^2-4A\kappa}}{2A} > 0$, then $\sigma^2 - 4A\kappa \geq 0, \sigma < 0$ and $-\sigma$
$-\sqrt{\sigma^2 - 4A\kappa} > 0 \Leftrightarrow -4A\kappa < 0$, such that $\kappa > 0 \Leftrightarrow AB > c_1$. Considering $\tilde{p} = \frac{-\sigma+\sqrt{\sigma^2-4A\kappa}}{2A} > 0$ then $\sigma^2 - 4A\kappa \geq 0, \sigma < 0$ and $-\sigma + \sqrt{\sigma^2 - 4A\kappa} > 0 \Leftrightarrow -4A\kappa > 0$, so that $\kappa < 0 \Leftrightarrow AB < c_1$. Therefore, the unique positive root exists for $\tilde{p} = \frac{-\sigma-\sqrt{\sigma^2-4A\kappa}}{2A} > 0$ with the conditions $\sigma^2 - 4A\kappa \geq 0, \sigma < 0$ and $AB > c_1$. So we obtain $T_1\left(\frac{\left(\frac{-\sigma-\sqrt{\sigma^2-4A\kappa}}{2A}\right)c_1}{B+\frac{\sigma+\sqrt{\sigma^2-4A\kappa}}{2A}}, \frac{-\sigma-\sqrt{\sigma^2-4A\kappa}}{2A}, 0\right)$ with the conditions $\sigma^2 - 4A\kappa \geq 0, \sigma < 0$ and $AB > c_1$.

Substituting $\sigma = -2AB - {c_1}^2 + c_1$ and $\kappa = AB^2 - Bc_1$ into the equation $\frac{\left(\frac{-\sigma-\sqrt{\sigma^2-4A\kappa}}{2A}\right)c_1}{B+\frac{\sigma+\sqrt{\sigma^2-4A\kappa}}{2A}}$ we get $\frac{\left(-(-2AB-{c_1}^2+c_1)-\sqrt{(-2AB-{c_1}^2+c_1)^2-4A(AB^2-Bc_1)}\right)c_1}{2AB+(-2AB-{c_1}^2+c_1)+\sqrt{(-2AB-{c_1}^2+c_1)^2-4A(AB^2-Bc_1)}} = \frac{-c_1-1+\sqrt{\Omega}}{2}$, with $\Omega = 4AB$
$+(c_1-1)^2$. Substituting $\sigma = -2AB - {c_1}^2 + c_1$ and $\kappa = AB^2 - Bc_1$ into the equation $\frac{-\sigma-\sqrt{\sigma^2-4A\kappa}}{2A}$ we obtain $\frac{-(-2AB-{c_1}^2+c_1)-\sqrt{(-2AB-{c_1}^2+c_1)^2-4A(AB^2-Bc_1)}}{2A} = \frac{2AB+c_1(c_1-1)-c_1\sqrt{\Omega}}{2A}$. So we get $T_1\left(\frac{-c_1-1+\sqrt{\Omega}}{2}, \frac{2AB+c_1(c_1-1)-c_1\sqrt{\Omega}}{2A}, 0\right)$where $\Omega = 4AB + (c_1-1)^2$ with the conditions $2AB$

$+c_1(c_1-1)>0$ and $AB>c_1$.

(2) Case $\tilde{p}=\frac{\mu h}{\beta-\mu}.\left(\frac{B\tilde{c}}{\tilde{c}+c_1}-\tilde{p}\right)\tilde{p}-\frac{\tilde{p}\tilde{z}}{\tilde{p}+h}=0 \Leftrightarrow \tilde{z}=\left(\frac{\mu h}{\beta-\mu}+h\right)\left(\frac{B\tilde{c}}{\tilde{c}+c_1}-\frac{\mu h}{\beta-\mu}\right)$.

Substituting $\tilde{p}=\frac{\mu h}{\beta-\mu}$ into the Eq. (33) we obtain $A\left(1-\frac{\tilde{c}}{\tilde{c}+1}\right)\tilde{p}-\tilde{c}=0 \Rightarrow \tilde{c}_{1,2}=\frac{-1\pm\sqrt{1+\frac{4A\mu h}{\beta-\mu}}}{2}$.

Then there is $T_2\left(\frac{-1+\sqrt{1+\frac{4A\mu h}{\beta-\mu}}}{2},\frac{\mu h}{\beta-\mu},\frac{\beta h\left(\mu B-\mu B\sqrt{1+\frac{4A\mu h}{\beta-\mu}}-\beta B+\beta B\sqrt{1+\frac{4A\mu h}{\beta-\mu}}+\mu h-\mu h\sqrt{1+\frac{4A\mu h}{\beta-\mu}}-2\mu h c_1\right)}{(\beta-\mu)^2\left(-1+\sqrt{1+\frac{4A\mu h}{\beta-\mu}}+2c_1\right)}\right)$

with the conditions $\beta>\mu$ and $\frac{B\left(-1+\sqrt{1+\frac{4A\mu h}{\beta-\mu}}\right)}{-1+\sqrt{1+\frac{4A\mu h}{\beta-\mu}}+2c_1}>\frac{\mu h}{\beta-\mu}$.

5.2STABILITY OF T_2 WITHOUT TIME DELAY

Theorem 2.

We have T_0, T_1 and T_2 which three equilibrium points of the Eqs. (16-18) as in theorem 1.

1. The equilibrium point T_0 is not informative.
2. The equilibrium point T_1 is node stable and node unstable.
3. The equilibrium point T_2 is node stable

Proof.

The general Jacobian matrix of Eqs. (27-29) is given by $J=\begin{pmatrix}\frac{-Ap}{(c+1)^2}-1 & \frac{A}{c+1} & 0\\ \frac{Bc_1p}{(c+c_1)^2} & \frac{Bc}{c+c_1}-2p-\frac{zh}{(p+h)^2} & -\frac{p}{p+h}\\ 0 & \frac{\beta hz}{(p+h)^2} & \frac{\beta p}{p+h}-\mu\end{pmatrix}$.

(1) At $T_0(0,0,0)$, the Jacobian matrix is $J(0,0,0)=\begin{pmatrix}-1 & A & 0\\ 0 & 0 & 0\\ 0 & 0 & -\mu\end{pmatrix}$. Thus, we get $\lambda_1=-1, \lambda_2=0$ and $\lambda_3=-\mu$.

(2) At $T_1(\tilde{c},\tilde{p},0)$, the Jacobian matrix is $J(\tilde{c},\tilde{p},0)=\begin{pmatrix}-\frac{A\tilde{p}}{(\tilde{c}+1)^2}-1-\lambda & \frac{A}{\tilde{c}+1} & 0\\ \frac{Bc_1\tilde{p}}{(\tilde{c}+c_1)^2} & \frac{B\tilde{c}}{\tilde{c}+c_1}-2\tilde{p}-\lambda & -\frac{\tilde{p}}{\tilde{p}+h}\\ 0 & 0 & \frac{\beta\tilde{p}}{\tilde{p}+h}-\mu-\lambda\end{pmatrix}$. Thus, we get $\lambda_1=\frac{\beta\tilde{p}}{\tilde{p}+h}-\mu \vee \lambda_{1,2}=\frac{-\kappa\pm\sqrt{\kappa^2-4\Theta}}{2}$ where $\kappa=\frac{A\tilde{p}}{(\tilde{c}+1)^2}-\frac{B\tilde{c}\gamma}{\tilde{c}+c_1}+2\tilde{p}+1$ and $\Theta=\frac{A\tilde{p}\left(-B\tilde{c}^2-2B\tilde{c}c_1+2\tilde{p}\tilde{c}^2+4\tilde{p}\tilde{c}c_1+2\tilde{p}{c_1}^2\right)}{(\tilde{c}+1)^2(\tilde{c}+c_1)^2}$
$+\frac{B\left(\tilde{c}^4-2\tilde{c}^3-\tilde{c}^2-\tilde{c}^3c_1-2\tilde{c}^2c_1-\tilde{c}c_1-Ac_1\tilde{p}\right)}{(\tilde{c}+1)^2(\tilde{c}+c_1)^2}+\frac{2\tilde{p}\left(\tilde{c}^4+2\tilde{c}^3c_1+\tilde{c}^2{c_1}^2+2\tilde{c}^3+4\tilde{c}^2c_1+2{c_1}^2\tilde{c}+\tilde{c}^2+2c_1\tilde{c}\right)}{(\tilde{c}+1)^2(\tilde{c}+c_1)^2}$.

(3) At $T_2(\tilde{c},\tilde{p},\tilde{z})$ the Jacobian matrix is $J(\tilde{c},\tilde{p},\tilde{z})=\begin{pmatrix}\frac{-A\tilde{p}}{(\tilde{c}+1)^2}-1 & \frac{A}{\tilde{c}+1} & 0\\ \frac{Bc_1\tilde{p}}{(\tilde{c}+c_1)^2} & \frac{B\tilde{c}}{\tilde{c}+c_1}-2\tilde{p}-\frac{\tilde{z}h}{(\tilde{p}+h)^2} & -\frac{\tilde{p}}{\tilde{p}+h}\\ 0 & \frac{\beta h\tilde{z}}{(\tilde{p}+h)^2} & \frac{\beta\tilde{p}}{\tilde{p}+h}-\mu\end{pmatrix}$
$=\lambda^3-\phi\lambda^2-\varepsilon\lambda-\rho=0$. If $\phi=-\mu-\frac{A\tilde{p}}{(\tilde{c}+1)^2}-1+\frac{B\tilde{c}}{\tilde{c}+c_1}-2\tilde{p}-\frac{\tilde{z}h}{(\tilde{p}+h)^2}+\frac{\beta\tilde{p}}{\tilde{p}+h}, \varepsilon=-\frac{A\tilde{p}\mu}{(\tilde{c}+1)^2}$
$-\mu+\frac{B\tilde{c}\mu}{\tilde{c}+c_1}-2\tilde{p}\mu-\frac{\tilde{z}h\mu}{(\tilde{p}+h)^2}+\frac{A\tilde{p}B\tilde{c}}{(\tilde{c}+1)^2(\tilde{c}+c_1)}-\frac{2\tilde{p}^2A}{(\tilde{c}+1)^2}-\frac{A\tilde{p}\tilde{z}h}{(\tilde{c}+1)^2(\tilde{p}+h)^2}+\frac{B\tilde{c}}{\tilde{c}+c_1}-2\tilde{p}-\frac{\tilde{z}h}{(\tilde{p}+h)^2}$
$+\frac{A\tilde{p}^2\beta}{(\tilde{p}+h)(\tilde{c}+1)^2}+\frac{\beta\tilde{p}}{\tilde{p}+h}-\frac{\beta\tilde{p}B\tilde{c}}{(\tilde{c}+c_1)(\tilde{p}+h)}+\frac{2\beta\tilde{p}^2}{\tilde{p}+h}+\frac{ABc_1\tilde{p}}{(\tilde{c}+1)(\tilde{c}+c_1)^2}, \rho=-\frac{A\beta\tilde{p}^2\tilde{z}h}{(\tilde{c}+1)^2(\tilde{p}+h)^3}-\frac{\beta\tilde{p}h\tilde{z}}{(\tilde{p}+h)^3}$

$$-\frac{AB\beta\tilde{p}^2\tilde{c}}{(\tilde{c}+1)^2(\tilde{c}+c_1)(\tilde{p}+h)}+\frac{2A\tilde{p}^3\beta}{(\tilde{p}+h)(\tilde{c}+1)^2}+\frac{A\beta\tilde{p}^2\tilde{z}h}{(\tilde{c}+1)^2(\tilde{p}+h)^3}-\frac{B\beta\tilde{p}\tilde{c}}{(\tilde{c}+c_1)(\tilde{p}+h)}+\frac{2\beta\tilde{p}^2}{\tilde{p}+h}+\frac{\beta\tilde{p}\tilde{z}h}{(\tilde{p}+h)^3}$$
$$-\frac{AB\beta\tilde{p}^2c_1}{(\tilde{c}+1)(\tilde{c}+c_1)^2(\tilde{p}+h)}+\frac{ABc_1\tilde{p}\mu}{(\tilde{c}+1)(\tilde{c}+c_1)^2}-\frac{2A\tilde{p}^2\mu}{(\tilde{c}+1)^2}-\frac{A\tilde{p}\tilde{z}h\mu}{(\tilde{c}+1)^2(\tilde{p}+h)^2}+\frac{B\tilde{c}\mu}{\tilde{c}+c_1}-2\tilde{p}\mu-\frac{\tilde{z}h\mu}{(\tilde{p}+h)^2}+\frac{AB\tilde{p}\tilde{c}\mu}{(\tilde{c}+1)^2(\tilde{c}+c_1)}.$$

So that, we have the eigenvalues $\lambda_1=\frac{\phi}{3}+\frac{2^{\frac{1}{3}}(-\phi^2-3\varepsilon)}{3\left(-27\rho-2\phi^3-9\phi\varepsilon+3\sqrt{3}\sqrt{27\rho^2+4\rho\phi^3+18\rho\phi\varepsilon-\phi^2\varepsilon^2-4\varepsilon^3}\right)^{\frac{1}{3}}}$
$-\frac{\left(-27\rho-2\phi^3-9\phi\varepsilon+3\sqrt{3}\sqrt{27\rho^2+4\rho\phi^3+18\rho\phi\varepsilon-\phi^2\varepsilon^2-4\varepsilon^3}\right)^{\frac{1}{3}}}{3.2^{\frac{1}{3}}}$, $\lambda_2=\frac{\phi}{3}-$
$\frac{(1+i\sqrt{3})(-\phi^2-3\varepsilon)}{3.2^{\frac{2}{3}}\left(-27\rho-2\phi^3-9\phi\varepsilon+3\sqrt{3}\sqrt{27\rho^2+4\rho\phi^3+18\rho\phi\varepsilon-\phi^2\varepsilon^2-4\varepsilon^3}\right)^{\frac{1}{3}}}$
$+\frac{(1-i\sqrt{3})\left(-27\rho-2\phi^3-9\phi\varepsilon+3\sqrt{3}\sqrt{27\rho^2+4\rho\phi^3+18\rho\phi\varepsilon-\phi^2\varepsilon^2-4\varepsilon^3}\right)^{\frac{1}{3}}}{6.2^{\frac{1}{3}}}$, $\lambda_3=\frac{\phi}{3}-$
$\frac{(1-i\sqrt{3})(-\phi^2-3\varepsilon)}{3.2^{\frac{2}{3}}\left(-27\rho-2\phi^3-9\phi\varepsilon+3\sqrt{3}\sqrt{27\rho^2+4\rho\phi^3+18\rho\phi\varepsilon-\phi^2\varepsilon^2-4\varepsilon^3}\right)^{\frac{1}{3}}}$
$+\frac{(1+i\sqrt{3})\left(-27\rho-2\phi^3-9\phi\varepsilon+3\sqrt{3}\sqrt{27\rho^2+4\rho\phi^3+18\rho\phi\varepsilon-\phi^2\varepsilon^2-4\varepsilon^3}\right)^{\frac{1}{3}}}{6.2^{\frac{1}{3}}}$.

5.3STABILITY OF T_2 WITH TIME DELAY

Theorem 3

Suppose that the conditions $\beta>\mu$ and $\frac{B\left(-1+\sqrt{1+\frac{4A\mu h}{\beta-\mu}}\right)}{-1+\sqrt{1+\frac{4A\mu h}{\beta-\mu}}+2c_1}>\frac{\mu h}{\beta-\mu}$ are satisfied and given τ_k and $\omega_k>0$ is obtained from the equation $\omega_k=\sqrt{z^*}$ where $z^3+Bz^2+Cz+D=0$ and $\omega^6+A\omega^4+B\omega^2+C=0$, with

$A=b_7{}^2+2b_5b_6+2b_2b_3+b_1{}^2-b_4{}^2$,
$B=2b_1{}^2b_5b_6+b_1{}^2b_7{}^2+b_2{}^2b_3{}^2+2b_2b_3b_5b_6+2b_2b_3b_7{}^2+b_5{}^2b_6{}^2-b_1{}^2b_4{}^2-b_4{}^2b_7{}^2$,
$C=b_1{}^2b_5{}^2b_6{}^2+2b_1b_2b_3b_5b_6b_7+b_2{}^2b_3{}^2b_7{}^2-b_1{}^2b_4{}^2b_7{}^2$

with the Routh Hurwitz conditions where

$b_1=\frac{-A\tilde{p}}{(\tilde{c}+1)^2}-1, b_2=\frac{A}{\tilde{c}+1}, b_3=\frac{Bc_1\tilde{p}}{(\tilde{c}+c_1)^2}, b_4=\frac{B\tilde{c}}{\tilde{c}+c_1}-2\tilde{p}-\frac{\tilde{z}h}{(\tilde{p}+h)^2}, b_5=-\frac{\tilde{p}}{\tilde{p}+h}, b_6=\frac{\beta h\tilde{z}}{(\tilde{p}+h)^2}$,
$b_7=\frac{\beta\tilde{p}}{\tilde{p}+h}-\mu$ furthermore T_2 is the equilibrium of Eqs. (16-18), then

1. The interior equilibrium T_2 of Eqs. (16-18) is stable when $\tau<\tau_k$ and unstable when $\tau>\tau_k$.
2. Eqs. (16-18) undergo a Hopf bifurcation at the interior equilibrium T_2 when $\tau=\tau_k$.

Proof.

The results of the analysis showed that the equilibrium point T_2 is stable. By giving the time-delay $\tau>0$ will cause the change in the stability of the equilibrium point T_2. To analyze the stability of the equilibrium point T_2 with time-delay, we linearize the model (27-29) around the equilibrium point T_2, then we obtain the linearized model

$$\frac{dc}{dt}=b_1c(t)+b_2p(t), \quad (36)$$
$$\frac{dp}{dt}=b_3c(t)+b_4p(t-\tau)+b_5z(t), \quad (37)$$
$$\frac{dz}{dt}=b_6p(t)+b_7z(t), \quad (38)$$

where $b_1=\frac{-A\tilde{p}}{(\tilde{c}+1)^2}-1, b_2=\frac{A}{\tilde{c}+1}, b_3=\frac{Bc_1\tilde{p}}{(\tilde{c}+c_1)^2}, b_4=\frac{B\tilde{c}}{\tilde{c}+c_1}-2\tilde{p}-\frac{\tilde{z}h}{(\tilde{p}+h)^2}, b_5=-\frac{\tilde{p}}{\tilde{p}+h}, b_6=\frac{\beta h\tilde{z}}{(\tilde{p}+h)^2}$,
$b_7=\frac{\beta\tilde{p}}{\tilde{p}+h}-\mu$.

Suppose the solution of Eqs. (36-38) is

$$c(t) = le^{\lambda\tau}, p(t) = me^{\lambda\tau}, z(t) = ne^{\lambda\tau}. \qquad (39)$$

Substituting Eq. (39) into Eqs. (36-38), then divided $e^{\lambda\tau}$ such that we get

$$l\lambda = b_1 l + b_2 m, \quad (40)$$
$$m\lambda = b_3 l + b_4 m + b_5 n, \quad (41)$$
$$n\lambda = b_6 m e^{-\lambda\tau} + b_7 n. \quad (42)$$

Eqs. (40-42) can be written in the following form.

$$\begin{bmatrix} l\lambda \\ m\lambda \\ n\lambda \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & 0 \\ b_3 & b_4 e^{-\lambda\tau} & b_5 \\ 0 & b_6 & b_7 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}.$$

So we get the following characteristic equation.

$$\begin{vmatrix} b_1 - \lambda & b_2 & 0 \\ b_3 & b_4 - \lambda & b_5 \\ 0 & b_6 e^{-\lambda\tau} & b_7 - \lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda^3 + \left(-b_1 - b_4 e^{-\lambda\tau} - b_7\right)\lambda^2 + \left(b_1 b_4 e^{-\lambda\tau} + b_1 b_7 + b_4 b_7 e^{-\lambda\tau} - b_2 b_3 - b_5 b_6\right)\lambda$$
$$-b_1 b_4 b_7 e^{-\lambda\tau} + b_2 b_3 b_7 + b_5 b_6 b_1 = 0$$

The eigenvalues of the characteristic equation (43) are either real and negative or complex conjugate with negative real parts if only if $-b_1 - b_4 - b_7 > 0, b_1 b_4 + b_1 b_7 + b_4 b_7 - b_2 b_3 - b_5 b_6 > 0$ and $-b_1 b_4 b_7 + b_2 b_3 b_7 + b_5 b_6 b_1 > 0$ [5]. So with the existence of time-delay, the equilibrium point T_2 is stable if and only if both conditions and are satisfied. Such that the eigenvalues of the equation (43) we let $\lambda = \mu \pm i\omega$ with $\mu = 0$ and $\omega > 0$ $(\lambda = \pm i\omega)$. To see the change in the stability of the equation model with time delay, then that eigenvalues are substituted into the equation (43) such that we obtain the roots of the characteristic equation

$$\Delta(i\omega,\tau) = b_1 b_4 \omega \sin\omega\tau + b_4 b_7 \omega \sin\omega\tau + b_4\omega^2 \cos\omega\tau + b_1\omega^2 + b_7\omega^2 - b_1 b_4 b_7 \cos\omega\tau + b_5 b_6 b_1 + b_2 b_3 b_7$$
$$+(-b_4\omega^2 \sin\omega\tau - \omega^3 + b_1 b_4 b_7 \sin\omega\tau + b_1 b_4 \omega \cos\omega\tau + b_4 b_7 \omega \cos\omega\tau + b_1 b_7 \omega - b_2 b_3 \omega - b_5 b_6 \omega)i$$

Equation (44) is zero if the imaginary and real part are zero, so we obtain

$$-\omega^3 + b_1 b_7 \omega - b_2 b_3 \omega - b_5 b_6 \omega = b_4 \omega^2 \sin\omega\tau - b_1 b_4 b_7 \sin\omega\tau - b_1 b_4 \omega \cos\omega\tau - b_4 b_7 \omega \cos\omega\tau$$

and

$$-b_1\omega^2 - b_7\omega^2 - b_5 b_6 b_1 - b_2 b_3 b_7 = b_1 b_4 \omega \sin\omega\tau + b_4 b_7 \omega \sin\omega\tau + b_4\omega^2 \cos\omega\tau - b_1 b_4 b_7 \cos\omega\tau.$$

Furthermore, eliminating Eqs. (45-46) to τ by Squaring both sides gives

$$b_1{}^2 b_7{}^2 \omega^2 - 2b_1 b_2 b_3 b_7 \omega^2 - 2b_1 b_5 b_6 b_7 \omega^2 - 2b_1 b_7 \omega^4 + b_2{}^2 b_3{}^2 \omega^2 + 2b_2 b_3 b_5 b_6 \omega^2 + 2b_2 b_3 \omega^4$$
$$+b_5{}^2 b_6{}^2 \omega^2 + 2b_5 b_6 \omega^4 + \omega^6 = b_1{}^2 b_4{}^2 b_7{}^2 \sin^2\omega\tau + 2b_1{}^2 b_4{}^2 b_7 \omega \cos\omega\tau \sin\omega\tau$$
$$+b_1{}^2 b_4{}^2 \omega^2 \cos^2\omega\tau + 2b_1 b_4{}^2 b_7{}^2 \omega \cos\omega\tau \sin\omega\tau + 2b_1 b_4{}^2 b_7 \omega^2 \cos^2\omega\tau - 2b_1 b_4{}^2 b_7 \omega^2 \sin^2\omega\tau$$
$$-2b_1 b_4{}^2 \omega^3 \cos\omega\tau \sin\omega\tau + b_4{}^2 b_7{}^2 \omega^2 \cos^2\omega\tau - 2b_4{}^2 b_7 \omega^3 \cos\omega\tau \sin\omega\tau + b_4{}^2 \omega^4 \sin^2\omega\tau$$

and

$$b_1{}^2 b_5{}^2 b_6{}^2 + 2b_1{}^2 b_5 b_6 \omega^2 + b_1{}^2 \omega^4 + 2b_1 b_2 b_3 b_5 b_6 b_7 + 2b_1 b_2 b_3 b_7 \omega^2 + 2b_1 b_5 b_6 b_7 \omega^2 + 2b_1 b_7 \omega^4$$
$$+b_2{}^2 b_3{}^2 b_7{}^2 + 2b_2 b_3 b_7{}^2 \omega^2 + b_7{}^2 \omega^4 = b_1{}^2 b_4{}^2 b_7{}^2 \cos^2\omega\tau - 2b_1{}^2 b_4{}^2 b_7 \omega \cos\varpi\tau \sin\varpi\tau$$
$$+b_1{}^2 b_4{}^2 \omega^2 \sin^2\omega\tau - 2b_1 b_4{}^2 b_7{}^2 \omega \cos\varpi\tau \sin\varpi\tau - 2b_1 b_4{}^2 b_7 \omega^2 \cos^2\omega\tau + 2b_1 b_4{}^2 b_7 \omega^2 \sin^2\omega\tau$$
$$+2b_1 b_4{}^2 \omega^3 \cos\varpi\tau \sin\varpi\tau + b_4{}^2 b_7{}^2 \omega^2 \sin^2\omega\tau + 2b_4{}^2 b_7 \omega^3 \cos\varpi\tau \sin\varpi\tau + b_4{}^2 \omega^4 \cos^2\omega\tau$$

Then adding both Eqs. (47-48) and regrouping by powers of ω, we obtain the following fourth degree polynomial

$$\omega^6 + A\omega^4 + B\omega^2 + C = 0,$$

with

$A = {b_7}^2 + 2b_5b_6 + 2b_2b_3 + {b_1}^2 - {b_4}^2,$
$B = 2{b_1}^2b_5b_6 + {b_1}^2{b_7}^2 + {b_2}^2{b_3}^2 + 2b_2b_3b_5b_6 + 2b_2b_3{b_7}^2 + {b_5}^2{b_6}^2 - {b_1}^2{b_4}^2 - {b_4}^2{b_7}^2,$
$C = {b_1}^2{b_5}^2{b_6}^2 + 2b_1b_2b_3b_5b_6b_7 + {b_2}^2{b_3}^2{b_7}^2 - {b_1}^2{b_4}^2{b_7}^2.$
To simplify the calculation suppose $z = \omega^2$, so Eq. (49) changes to

$$z^3 + Az^2 + Bz + C = 0, \qquad (50)$$

the root value of equation (50) is determined by Lemma 1 as follows.

Lemma 1. [9]
Define $\xi = A^2 - 3C$.
(i) If $C < 0$, then equation (50) has unique simple positive root.
(ii) If $C \geq 0$ and $\xi < 0$, then equation (50) does not have real roots.
(iii) If $C \geq 0$ and $\xi \geq 0$, then the equation (50) has two possitive roots if only if $z = \frac{1}{3}\left(-A + \sqrt{\xi}\right) > 0$ and $h(z) \leq 0$.

Suppose that equation (50) has simple positive roots. Without loss of generality, we assume that it has three positive roots, denoted by z_1, z_2 and z_3 respectively. Then equation (50) has three positive roots, say $\omega_1 = \sqrt{z_1}, \omega_2 = \sqrt{z_2}$ and $\omega_3 = \sqrt{z_3}$.
Furthermore substituting ω_k into the Eqs. (45-46) and solving for τ_k, we get

$$\tau_k = \frac{1}{\omega_k} tan^{-1}\left(\frac{{b_1}^2b_4b_5b_6\omega_k + {b_1}^2b_4{b_7}^2\omega_k + {b_1}^2b_4{\omega_k}^3 + b_2b_3b_4{b_7}^2\omega_k + b_2b_3b_4{b_7}^2\omega_k + b_2b_3b_4{\omega_k}^3 + b_4b_5b_6{\omega_k}^3 + b_4{b_7}^2{\omega_k}^3 + b_4{\omega_k}^5}{-{b_1}^2b_4b_5b_6b_7 - b_1b_2b_3b_4{b_7}^2 - b_1b_2b_3b_4{\omega_k}^2 - b_4b_5b_6b_7{\omega_k}^2}\right) + \frac{2k\pi}{\omega_k},$$
$$k = 0, 1, 2, \cdots \qquad (51)$$

Lemma 2
If one of the following is true.
(1) $C < 0$ and $h'\left(\omega_{bif}\right) \neq 0$;
(2) $C \geq 0, \xi > 0, \bar{z} > 0$ and $h'(\bar{z}) < 0$;

then $\frac{dRe\lambda\left(\tau_{bif}\right)}{d\tau} \neq 0$,
where τ_{bif} and ω_{bif} defined in Eq. (50).
Further differentiating the Eq. (43) to τ, then we obtain

$$\lambda^3 + P\lambda^2 + Q\lambda + R + (S\lambda^2 + T\lambda + U)e^{-\lambda\tau} = 0$$
$$P = -b_1 - b_7$$
$$Q = b_1b_7 - b_2b_3 - b_5b_6$$
$$R = b_2b_3b_7 + b_5b_6b_1$$
$$S = -b_4$$
$$T = b_1b_4 + b_4b_7$$
$$U = -b_1b_4b_7$$

$$\lambda^3 + P\lambda^2 + Q\lambda + R + S\lambda^2e^{-\lambda\tau} + T\lambda e^{-\lambda\tau} + Ue^{-\lambda\tau} = 0$$
$$\Leftrightarrow \frac{d(\lambda^3)}{d\lambda}\frac{d\lambda}{d\tau} + P\frac{d(\lambda^2)}{d\lambda}\frac{d\lambda}{d\tau} + Q\frac{d\lambda}{d\lambda}\frac{d\lambda}{d\tau} + S\left\{\lambda^2\left[\frac{d(e^{-\lambda\tau})}{d(-\lambda\tau)}\left(-\lambda.1 + \tau\frac{d(-\lambda)}{d\lambda}\frac{d\lambda}{d\tau}\right)\right] + e^{-\lambda\tau}\frac{d(\lambda^2)}{d\lambda}\frac{d\lambda}{d\tau}\right\} +$$
$$T\left\{\lambda\left[\frac{d(e^{-\lambda\tau})}{d(-\lambda\tau)}\left(-\lambda.1 + \tau\frac{d(-\lambda)}{d\lambda}\frac{d\lambda}{d\tau}\right)\right] + e^{-\lambda\tau}\frac{d\lambda}{d\lambda}\frac{d\lambda}{d\tau}\right\} + U\left[\frac{d(e^{-\lambda\tau})}{d(-\lambda\tau)}\left(-\lambda.1 + \tau\frac{d(-\lambda)}{d\lambda}\frac{d\lambda}{d\tau}\right)\right] = 0$$
$$\Leftrightarrow 3\lambda^2\frac{d\lambda}{d\tau} + 2P\lambda\frac{d\lambda}{d\tau} + Q\frac{d\lambda}{d\tau} - S\lambda^3e^{-\lambda\tau} - S\lambda^2\tau\frac{d\lambda}{d\tau}e^{-\lambda\tau} + 2S\lambda\frac{d\lambda}{d\tau}e^{-\lambda\tau} - T\lambda^2e^{-\lambda\tau} -$$
$$T\lambda\tau\frac{d\lambda}{d\tau}e^{-\lambda\tau} + T\frac{d\lambda}{d\tau}e^{-\lambda\tau} - U\lambda e^{-\lambda\tau} - U\tau\frac{d\lambda}{d\tau}e^{-\lambda\tau} = 0$$
$$\Leftrightarrow \frac{d\lambda}{d\tau} = \frac{(S\lambda^2 + T\lambda + U)\lambda e^{-\lambda\tau}}{3\lambda^2 + 2P\lambda + Q - (S\lambda^2 + T\lambda + U)\tau e^{-\lambda\tau} + (2S\lambda + T)e^{-\lambda\tau}}.$$

From equation (52), we have $e^{-\lambda\tau} = \frac{-\lambda^3 - P\lambda^2 - Q\lambda}{S\lambda^2 + T\lambda + U}$. Then we get

$\frac{d\lambda}{d\tau} = \frac{\lambda(-\lambda^3 - P\lambda^2 - Q\lambda)}{3\lambda^2 + 2P\lambda + Q - \tau(-\lambda^3 - P\lambda^2 - Q\lambda) + (2S\lambda + T)e^{-\lambda\tau}}.$

$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau_b}$

$= \frac{\lambda(-\lambda^3 - P\lambda^2 - Q\lambda)}{3\lambda^2 + 2P\lambda + Q - \tau(-\lambda^3 - P\lambda^2 - Q\lambda) + (2S\lambda + T)e^{-\lambda\tau}}$

$= \frac{i\omega_b(-(i\omega_b)^3 - P(i\omega_b)^2 - Qi\omega_b)}{3(i\omega_b)^2 + 2Pi\omega_b + Q - \tau_b(-(i\omega_b)^3 - P(i\omega_b)^2 - Qi\omega_b) + (2Si\omega_b + S)(\cos\omega_b\tau_b - i\sin\omega_b\tau_b)}$

$= \frac{-\omega_b{}^4 + iJ\omega_b{}^3 + K\omega_b{}^2}{-3\omega_b{}^2 + 2Pi\omega_b + Q - \tau_b i\omega_b{}^3 - \tau_b P\omega_b{}^2 + \tau_b Qi\omega_b + 2Si\omega_b\cos\omega_b\tau_b + 2S\omega_b\sin\omega_b\tau_b + T\cos\omega_b\tau_b - Ti\sin\omega_b\tau_b}$

$= \frac{-\omega_b{}^4 + Q\omega_b{}^2 + P\omega_b{}^3 i}{P_1{}^2 + Q_1{}^2} . (P_1 - Q_1 i)$

with

$P_1 = -3\omega_b{}^2 + Q - \tau_b P\omega_b{}^2 + 2S\omega_b\sin\omega_b\tau_b + T\cos\omega_b\tau_b$

$Q_1 = 2P\omega_b - \tau_b\omega_b{}^3 + \tau_b Q\omega_b + 2S\omega_b\cos\omega_b\tau_b - S\sin\omega_b\tau_b$

$= \frac{3\omega_b{}^6 - 4Q\omega_b{}^4 + 2P^2\omega_b{}^4 + Q^2\omega_b 2}{P_1{}^2 + Q_1{}^2}$

Hence $\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_b} = \frac{\omega_b{}^2(3\omega_b{}^4 + (-4Q + 2P^2)\omega_b{}^2)}{P_1{}^2 + Q_1{}^2} \neq 0$

5.4SIMULATIONS AT T_2 WITH TIME DELAY

The numerical simulations of the phytoplankton-zooplankton-dissolved oxygen model using Holling II with time delay performed to show the effect of time delay on the equilibrium point T_2 stability. The parameter values used for the simulations at equilibrium point T_2 with time delay presented as following.

$B = 1;\ \beta = 2;\ \mu = 0{,}6;\ h = 0{,}35;\ A = 1; c_1 = 0{,}01.$ Hence obtained

$\tilde{c} = \frac{-1 + \sqrt{1 + \frac{4A\mu h}{\beta - \mu}}}{2} = \frac{-1 + \sqrt{1 + \frac{4.0{,}53.0{,}5.0{,}1}{1 - 0{,}5}}}{2} = 0{,}647508942.$

$\tilde{p} = \frac{\mu h}{\beta - \mu} = \frac{0{,}5.0{,}1}{1 - 0{,}5} = 0{,}15.$

$\tilde{z} = \frac{\beta h(-\mu B\tilde{c} + \beta B\tilde{c} - \mu h\tilde{c} - \mu h c_1)}{(\beta - \mu)^2(\tilde{c} + c_1)} = 0{,}4173955408$

So obtained the equilibrium point $T_2(0{,}647508942; 0{,}15; 0{,}4173955408)$.

Then from the parameter values presented in Table 1, obtained

$b_1 = \frac{-A\tilde{p}}{(\tilde{c}+1)^2} - 1 = -1{,}055263158.$

$b_2 = \frac{A}{\tilde{c}+1} = 0{,}6069769787.$

$b_3 = \frac{Bc_1\tilde{p}}{(\tilde{c}+c_1)^2} = 0{,}003469668090.$

$b_4 = \frac{B\tilde{c}}{\tilde{c}+c_1} - 2\tilde{p} - \frac{\tilde{z}h}{(\tilde{p}+h)^2} = 0{,}1004373242.$

$b_5 = -\frac{\tilde{p}}{\tilde{p}+h} = -0{,}3.$

$b_6 = \frac{\beta h\tilde{z}}{(\tilde{p}+h)^2} = 1{,}168707514.$

$b_7 = \frac{\beta\tilde{p}}{\tilde{p}+h} - \mu = \frac{1.0{,}1}{0{,}1+0{,}1} - 0{,}5 = 0.$

$\omega^6 + (b_7{}^2 + 2b_5b_6 + 2b_2b_3 + b_1{}^2 - b_4{}^2)\omega^4 + (2b_1{}^2b_5b_6 + b_1{}^2b_7{}^2 + b_2{}^2b_3{}^2 + 2b_2b_3b_5b_6 + 2b_2b_3b_7{}^2 + b_5{}^2b_6{}^2 - b_1{}^2b_4{}^2 - b_4{}^2b_7{}^2)\omega^2 + b_1{}^2b_5{}^2b_6{}^2 + 2b_1b_2b_3b_5b_6b_7 + b_2{}^2b_3{}^2b_7{}^2 - b_1{}^2b_4{}^2b_7{}^2 = 0.$

$\Leftrightarrow \omega^6 + 0{,}4064801858\omega^4 - 0{,}6693727703\omega^2 + 0{,}1368912641$

Because of $\omega > 0$, then selected $\omega_k = 0{,}641685611351827$.

Then searched the value of τ_k by substituting the values $b_1, b_2, b_3, b_4, b_5, b_6, b_7$ and ω_k into the following equation.

$$\tau_k = \frac{1}{\omega_k} tan^{-1}\left(\frac{-b_4\omega_k{}^3 - b_7\omega_k{}^3 - b_2b_3b_7\omega_k - b_1{}^2b_4\omega_k - b_1{}^2b_7\omega_k + b_1b_2b_3\omega_k}{b_1{}^2\omega_k{}^2 + b_1b_2b_3b_7 - b_1{}^2b_4b_7 + \omega_k{}^4 - \omega_k{}^2b_4b_7 + b_2b_3\omega_k{}^2}\right) + \frac{2k\pi}{\omega_k}, k = 0, 1, 2, \cdots$$

$\Leftrightarrow \tau_k = 1{,}55.$

In this article only discussed the value of time delay when, before and after the delay timeout value in the distance $k = 0$.

In addition to the parameters mentioned in Table 3, it is necessary to select the time delay parameters indicated to show changes in equilibrium point stability. In this simulation will be provided three cases to indicate the existence of Hopf bifurcation.

Table 5.4.1 Selection of time delay and model stability

No	Case	τ	Equilibrium Point Stability
1	$\tau < \tau_k$	1,3	stable spiral
2	$\tau = \tau_k$	1,55	stable spiral
2	$\tau > \tau_k$	1,7	Unstable spiral

Simulation at T_2 for the case $\tau < \tau_k$

In this case, the parameter value used is $\tau = 1{,}3 < \tau_k$. The simulation results can be seen in Figure 3.

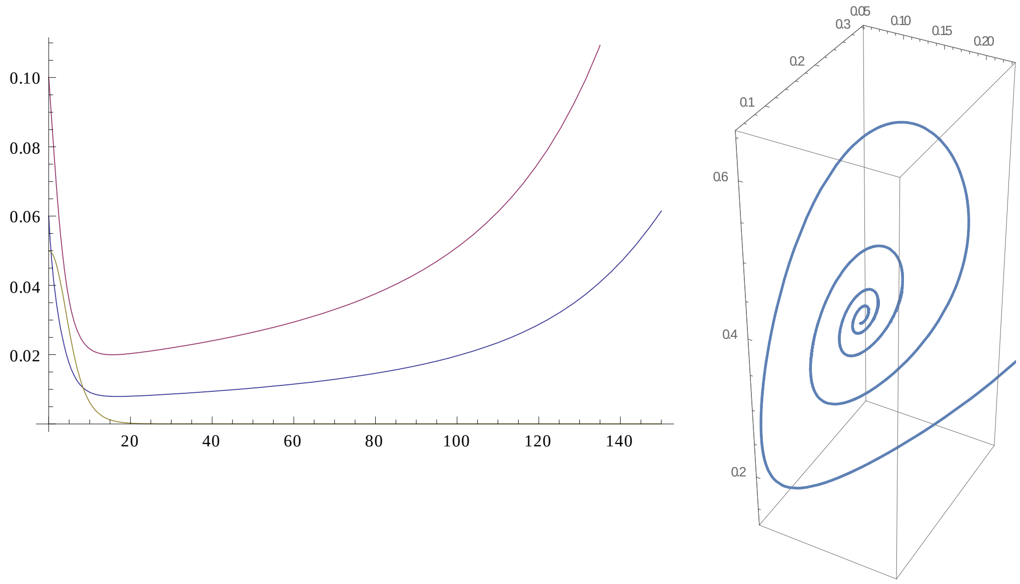


Figure 3. The solution field potret phase of predator-prey system at the equilibrium point T_2 in the case $\tau < \tau_k$

The Solution field shows that there are oscillations with increasingly smaller deviations, so that the oxygen-phyto-zooplankton develops and eventually stabilizes to a certain value.

Figure 3 also shows that point T_2 in the case $\tau < \tau_k$ is stable. So in this condition there is stability in the amount of oxygen-phyto-zooplankton.

Simulation at T_2 for the case $\tau = \tau_k$

In this case, the parameter value used is $\tau = 1{,}55 = \tau_k$. The simulation results can be seen in Figure 4.

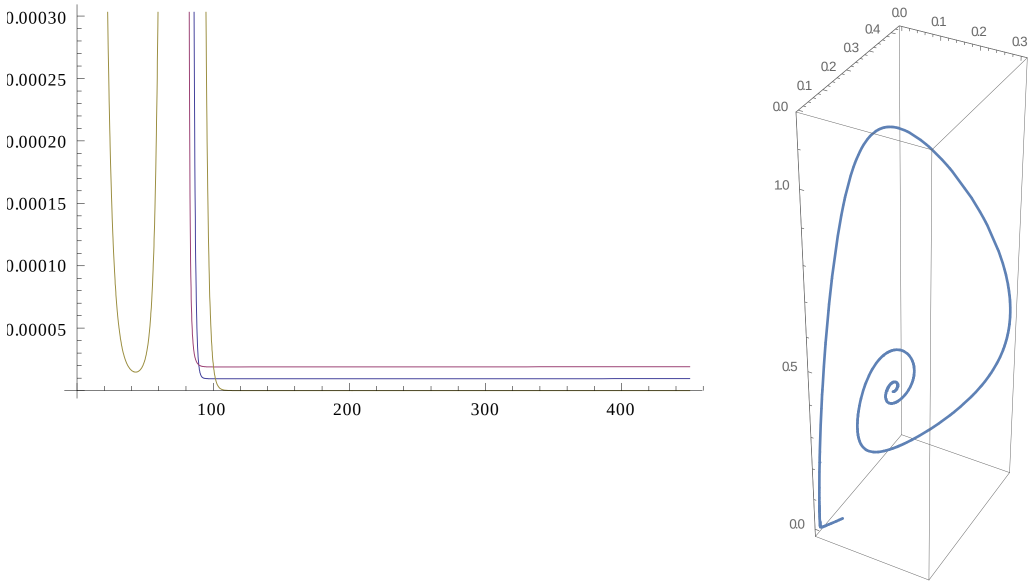


Figure 4. The solution field potret phase of predator-prey system at the equilibrium point T_2 in the case $\tau = \tau_k$

For the solution field, the system shows oscillations with increasingly smaller deviations, so that the oxygen-phyto-zooplankton develops and eventually stabilizes to a certain value.

Figure 4 also shows that T_2 is stable. So in this condition there is stability in the amount of oxygen-phyto-zooplankton.

Simulation at T_2 for the case $\tau > \tau_k$

In this case, the parameter value used is $\tau = 1{,}7 > \tau_k$. The simulation results can be seen in Figure 5.

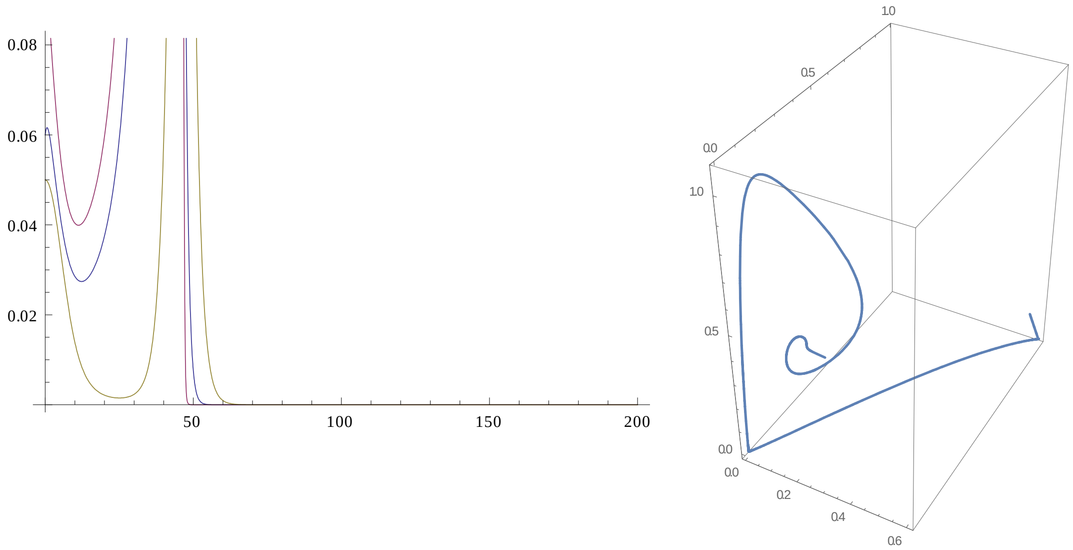


Figure 5. The solution field potret phase of predator-prey system at the equilibrium point T_2 in the case $\tau > \tau_k$

Figure 5 shows that there are oscillations with increasingly large deviations, so that the oxygen-phyto-zooplankton develops and at the end of time the oxygen-phyto-zooplankton decreases in number and almost becomes extinct. Figure 4 shows that T_2 is unstable. So in this condition there is no stability in the amount of oxygen-phyto-zooplankton.

6. CONCLUSIONS

From the above discussion, we can be concluded that based on the non-dimensional model, we obtain the following predator-prey model using Holling type II functional response with the time delay in a facultative waste stabilization pond

To analyze the existence of Hopf bifurcation, the predator-prey population dynamics was simulated based on three cases, by increasing the time-delay in the growth rate of the predator population (τ_k). By choosing an exact parameter value (τ_k), we can show the existence of Hopf bifurcation. In the case $\tau = \tau_k$ the stable spiral changed into an unstable spiral and also observed the presence of limit cycles. This is known as Hopf bifurcation. Then, to illustrate the model, simulation model was carried out using the Maple 12 software and mathematica 9. The model simulations gave the same result with the analysis.

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