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Gutman, Sombor, and Harmonic Indices of Unit Graphs in The Integer Ring Modulo with A Specific Order

Indeks Gutman, Sombor, dan Harmonik dari Graf Unit di Gelanggang Bilangan Bulat Modulo dengan Orde Spesifik

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Abstract

The ring of integers modulo n is an algebraic structure that plays an important role in various fields, such as number theory, cryptography, and number system modeling. This structure also has a strong connection to graph representation, especially in the formation of unit graphs. This research focuses on the analysis of unit graphs formed from modulo integer algebras of a certain order, which aims to formulate the general form of topological indices, namely Gutman, Sombor, Reduced Sombor, Average Sombor, and Harmonic. The research considers two cases: the ring of integer modulo $n = 2^k, k \in \mathbb{N}$ and $n = p$, where p is an odd prime number. The results show that each index has a unique and specific mathematical pattern according to its unit graph order. These findings provide a deeper understanding of the topological and combinatorial properties of unit graphs, which may help in generalizing their topological indices.

Keywords: Modulo Integer Ring, Unit Graph, Gutman Index, Sombor Index, Harmonic Index.

Abstrak

Gelanggang bilangan bulat modulo n merupakan salah satu struktur aljabar yang memiliki peran penting dalam berbagai bidang, seperti teori bilangan, kriptografi, dan pemodelan sistem bilangan. Struktur ini juga memiliki hubungan erat dengan representasi graf, khususnya dalam pembentukan graf unit. Penelitian ini berfokus pada analisis graf unit yang terbentuk dari gelanggang bilangan bulat modulo dengan orde tertentu, yang bertujuan untuk merumuskan bentuk umum dari indeks topologi, yaitu Gutman, Sombor, Sombor Tereduksi, Sombor Rata-rata, dan Harmonik. Penelitian dilakukan pada dua kasus, yaitu gelanggang bilangan bulat modulo dengan orde $n = 2^k, k \in \mathbb{N}$ dan

$n = p$, di mana p adalah bilangan prima ganjil. Hasil penelitian menunjukkan bahwa masing-masing indeks memiliki pola matematis yang unik dan spesifik sesuai dengan ordo graf unitnya. Temuan ini memberikan pemahaman lebih mendalam mengenai sifat-sifat topologis dan kombinatoris graf unit, yang dapat mendukung beberapa bentuk perumuman rumus dari indeks topologinya.

Kata kunci: Gelanggang Bilangan Bulat Modulo, Graf Unit, Indeks Gutman, Indeks Sombor, Indeks Harmonik.

1. INTRODUCTION

Modulo Integer Algebras have wide applications in number theory, cryptography, and number system modeling in computing. The study of this algebra has grown with the increasing need to understand the properties of elements and operations that may occur in modular arithmetic systems [3]. In addition, the structure of the ring is also related to the graph, where the vertices in the graph are connected based on modulo operations on the elements of the ring [5].

Graphs are sets of vertices connected by edges, which are used to model relationships between elements or models in various contexts [1]. One of the interesting models that emerged from the study of modulo integer algebras is the graph representation of the unit graph formed from the structure of the algebra. Some previous studies related to the topic of graph representation of ring theory include zero divisor graphs [16] and nilpotent graphs [12].

One of the concepts in graphs is the topological index, which is a numerical measure used to describe the characteristics of a graph. A topological index is a measure used to assess the characteristics of a graph, such as the relationship between vertices, or other specific properties related to the graph structure [14]. This metric plays an important role in graph analysis because it can measure and compare various aspects of a graph, such as how closely vertices neighbor each other and the distribution of vertices based on certain criteria.

Research on unit graphs in modulo integer algebras of a certain order attracts attention because it can reveal topological and combinatorial properties of the elements of the algebra. Therefore, researchers are interested in discussing the Gutman, Sombor, Reduced Sombor, Average Sombor, and Harmonic Indices on the unit graph formed from the modulo integer ring of a certain order. The choice of a particular order in the unit graph is to identify specific patterns and unique structures in the unit graph, which may not appear in graphs of more general order [13] .

Research on unit graphs in modulo integer algebras of a certain order attracts attention because it can reveal topological and combinatoric properties of the algebraic elements. Therefore, researchers are interested in discussing the Gutman, Sombor, Reduced Sombor, Average Sombor, and Harmonic Indices on the unit graph formed from modulo integer algebras of a certain order. The choice of a particular order on the unit graph is to identify special patterns and unique structures on the unit graph that may not appear on graphs with more general orders. The orders used in this research are $n = 2^k, k \in \mathbb{N}$ and $n = p$ where p is an odd prime number.

2. LITERATURE REVIEW

In this section, some supporting theories of research such as ring, graph and some indices used in research are explained.

2.1 Relevant Research

Several related studies have been conducted in this area. Research conducted [12] studied nilpotent graphs on the integer ring modulo $(\mathbb{Z}_n)$ of order that is a power of a prime number. The results show that if n is a prime number, then the nilpotent graph is a star graph $\left(K_{1,n-1}\right)$. Research conducted by [13] discussed the Sombor Index as a new graph invariant in graph theory, focusing on identifying graphs that have maximum and minimum Sombor Index among various classes of cyclic graphs. The results show that the cycle graph C_n has the minimum Sombor Index for $n \geq 4$ while the graph $H_{n,v}$ has the maximum Sombor Index in the cyclic v graph with $5 \leq v \leq n-2$. Research conducted by [18] and [17] focused on the formulation of Sombor, Reduced Sombor, and Average Sombor Indices on nilpotent graphs of modulo integer algebras and coprime graphs of quaternion groups. The results of this study show that there is a general formula that can be used to analyze the topological characteristics of a graph. Meanwhile, [9] discussed the topological index of the line graph derived from the identity graph in the integer multiplication group modulo a prime order greater than three is $\frac{p-3}{2}$. This research shows that the graph has certain subgraph structures, such as the C_3 subgraph, pendant vertex, complete subgraph K_{p-2}, and $\frac{p-3}{2}$ star subgraph $K_{1,2}$. In addition, this study succeeded in formulating the general form of several topological indices, including the Gutman and Harmonic indices. Research conducted by [10] examined the characteristics and properties of unit graphs in the integer ring modulo n. The results show that if n is a power of 2, then the unit graph forms a complete bipartite graph, while if n is a prime number, the unit graph forms a complete partite graph with $2n+1$ parts. In addition, this study successfully identifies various numerical invariants of the unit graph, such as vertex degree, chromatic number, and graph diameter, where the 0 vertex has degree $n-1$, other vertices have degree $n-2$, and the graph diameter is 2, which indicates that each vertex can be reached in at most two steps.

2.2 Ring

Definition 2.2.1 Suppose R is a nonempty set. R with binary operations of addition and multiplication, denoted by $(+)$ and $(\cdot)$, is called a ring if $\forall a, b, c \in R$ satisfies the following axioms [8].

1. The set R with addition operation $(R, +)$ is closed, hence $a + b \in R$.
2. The set $(R, +)$ is associative, hence $a + (b + c) = (a + b) + c$.
3. The set $(R, +)$ contains the identity element. There is $0 \in R$, hence $a + 0 = 0 + a = a$.
4. The set $(R, +)$ has an inverse element. There is $-a \in R$, so $a + (-a) = (-a) + a = 0$.
5. The set $(R, +)$ is commutative, hence $a + b = b + a$.
6. The set R with multiplication operation $(R,\cdot)$ is closed, hence $a \cdot b \in R$.
7. The set $(R,\cdot)$ is associative, hence $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
8. The set $(R,\cdot)$ is distributive to the addition operation, hence $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

Definition 2.2.2 Suppose R is a ring with identity element $x \in R$. An element u is called a unit on R if there exists $y \in R$ and an identity element $e \in R$ such that $x \cdot y = y \cdot x = e$. The set of all units on R is expressed as $U(R)$ [2].

2.3 Graph

Definition 2.3.1 A graph, or denoted by G, is a set of pairs $G = (V, E)$, where V is a nonempty set of vertices and E is a set of edges connecting a pair of vertices [6].

Definition 2.3.2 Order is defined by the number of points of G, denoted by $|G|$. While the size of G is defined by the number of edges of G, denoted by $|(|G|)|$ [6].

Definition 2.3.3 The unit graph of R denoted by $G(R)$, has the same set of vertices as the set of all elements of R, different vertices x and y are neighbors if and only if $x + y$ is a unit of R [2].

Definition 2.3.4 Suppose G is a graph, $P = \{v_1, v_2, \dots, v_n\} \subseteq V(G)$ is said to be a path if its vertices are neighboring [6].

Definition 2.3.5 The distance between two points u and v is the length of the shortest path $u - v$. If there is no shortest path of all $u - v$ paths in G. Then, $d_G(u, v) \coloneqq \infty$ is defined [6].

Definition 2.3.6 The degree of a point $v \in V(G)$, denoted by $deg_G(v)$, is defined as the number of edges adjacent to point v (or the number of points neighboring point v) [6].

Theorem 2.1.1 Suppose $\mathbb{Z}_n$ is a modulo integer ring of order $n = 2^k$. Then the unit graph $G(\mathbb{Z}_n)$ is a complete bipartite graph.

Theorem 2.1.2 Suppose $\mathbb{Z}_n$ is a modulo integer ring of order $n = p$, p is an odd prime number. Then, the unit graph $G(\mathbb{Z}_n)$ is a complete $\frac{n+1}{2} -$ partite graph.

2.4 Index Topology

A topological index is a concept for describing the structural characteristics of a graph without regard to size or geometric shape. Graphs are used to model compounds and chemical reactions in compounds. Topological indices as molecular descriptors to evaluate toxicity and to predict biological activity [7].

2.4.1 Gutman Index

Definition 2.4.1 Suppose there is a graph G. According to [4], The Gutman Index of a graph G is defined as follows

$$Gut(G) = \sum_{u,v \in V(G)} deg(u)deg(v)d(u,v)$$

where $deg(u)$ and $deg(v)$ are the degrees of vertex u and vertex v in graph G, while $d(u, v)$ is the distance from vertex u to vertex v in graph G.

2.4.2 Sombor Index

Definition 2.4.2 Suppose a graph G is a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The spacing index, reduced spacing index, and average spacing index of the graph G are as follows

$$SO(G) = \sum_{uv \in E(G)} \sqrt{deg(u)^2 + deg(v)^2}$$

$$SO_{red}(G) = \sum_{uv \in E(G)} \sqrt{(deg(u) - 1)^2 + (deg(v) - 1)^2}$$

$$SO_{avg}(G) = \sum_{uv \in E(G)} \sqrt{\left(\deg(u) - \frac{2m}{n}\right)^2 + \left(\deg(v) - \frac{2m}{n}\right)^2}$$

where $u, v \in V(G)$ and u and v are neighbors with $deg(u)$ and $deg(v)$ being the degrees of vertices u and v. Term m denotes the number of edges in graph G and n is the number of vertices in graph G [15].

2.4.3 Harmonic Index

Definition 2.4.3 Suppose a graph G. According to [11], The Harmonic Index of the graph G is defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{deg(u) + deg\,(v)}$$

where $deg(u)$ is the degree of vertex u is. the number of vertices $u \neq v$ that are neighbors of vertex u.

3. METHOD

This research is a theoretical descriptive and explorative research. This research aims to describe and analyze the characteristics of unit graphs and formulate certain topological indices in the context of modulo integer ring. The research procedures that carried out in this research are listed below.

1. Literature study by collecting and studying references related to unit graphs, modulo integer ranges, and topological indices such as Gutman, Sombor, Reduced Sombor, Average Sombor, and Harmonic.
2. Analyze the characteristics of unit graphs by analyzing the shape of unit graphs on modulo integer ranges with specific orders, as well as the characteristics of unit graphs.
3. Construct conjectures about the relationships and patterns of the Gutman, Sombor, Reduced Sombor, Average Sombor, and Harmonic Indices on unit graphs in the modulo integer ranges of specific order.
4. Proving the conjecture that has been developed. If the conjecture is proven, then the conjecture is stated as a theorem. If the conjecture is not proven, go back to the previous step to develop a new conjecture and prove it.
5. Draw conclusions from the research that has been done based on the results of the proof of the conjecture and the analysis carried out.

4. MAIN RESULT

4.1 Gutman, Sombor, Reduced Sombor, Average Sombor, and Harmonics Indices on Modulo Integer Fields of order $n = 2^k$

Theorem 4.1.1 Suppose $G(\mathbb{Z}_n)$ is a unit graph of $\mathbb{Z}_n$, with $n = 2^k$ for any $k \in \mathbb{N}$. Then, the Gutman Index of the unit graph $G(\mathbb{Z}_n)$ is $3\left(2^{k-1}\right)^4 - 2\left(2^{k-1}\right)^3$.

Proof. Based on **Theorem 2.1.1** $G(\mathbb{Z}_n)$ a complete bipartite graph. Then Lestari et all found that exist $V_1, V_2 \subset V$, where all vertices in V_1 are neighbors to all vertices in V_2 and no edges are neighbors in the same partition. As a result, there are two cases that need to be solved to find out the Gutman index of $G(\mathbb{Z}_n)$ as follows.

Case 1. For $u \in V_1$and $v \in V_2$, we get

$$\sum_{\substack{u \in V_1 \\ v \in V_2}} deg(u)deg(v)d(u,v) = \left(2^{k-1}\right)^2\left(2^{k-1}\right)^2(1)$$

$$= \left(2^{k-1}\right)^4.$$

Case 2. For $u, v \in V_i$, for $i = 1,2$ and $u \neq v$, we get

$$\sum_{i=1}^{2}\sum_{\substack{u,v\in V_i\\u\neq v}} deg(u)deg(v)d(u,v) = 2\binom{2^{k-1}}{2}\left(2^{k-1}\right)^2(2)$$
$$= 2\left(\frac{(2^{k-1})(2^{k-1}-1)(2^{k-1}-2)!}{(2)(2^{k-1}-2)!}\right)\left(2^{k-1}\right)^2(2)$$
$$= \left(2^{k-1}\right)\left(2^{k-1}-1\right)\left(2^{k-1}\right)^2(2)$$
$$= \left(2^{k-1}\right)^3\left(2^{k-1}-1\right)(2)$$
$$= 2\left(\left(2^{k-1}\right)^4-\left(2^{k-1}\right)^3\right).$$

Based on **Definition 2.4.1** and the two cases above, the Gutman Index of the unit graph $G(\mathbb{Z}_n)$ of degree $n = 2^k$ for a $k \in \mathbb{N}$ is

$$Gut\big(\mathrm{G}(R)\big) = \sum_{\substack{u\in V_1\\v\in V_2}} deg(u)deg(v)d(u,v) + \sum_{i=1}^{2}\sum_{\substack{u,v\in V_i\\u\neq v}} deg(u)deg(v)d(u,v)$$
$$= \left(2^{k-1}\right)^4 + 2\left(\left(2^{k-1}\right)^4-\left(2^{k-1}\right)^3\right)$$
$$= 3\left(2^{k-1}\right)^4 - 2\left(2^{k-1}\right)^3. \blacksquare$$

Theorem 4.1.2 Suppose let $G(\mathbb{Z}_n)$ is a unit graph of $\mathbb{Z}_n$, with $n = 2^k$ for any $k \in \mathbb{N}$. Then, The Spacing Sombor Index of the unit graph $G(\mathbb{Z}_n)$ is $\left(2^{k-1}\right)^3\sqrt{2}$.

Proof. Based on **Theorem 2.1.1** $G(\mathbb{Z}_n)$ a complete bipartite graph. Then Lestari et all found that there are $e_1, e_2 \in E$, and all vertices in V_1 are neighbors with all vertices in V_2 where $V_1, V_2 \subset V$. As a result, we obtain the solution based on **Definition 2.4.2** to find out the spacing index of $G(\mathbb{Z}_n)$ as follows.

$$SO\big(\mathrm{G}(R)\big) = \left(2^{k-1}\right)^2\sqrt{(2^{k-1})^2+(2^{k-1})^2}$$
$$= \left(2^{k-1}\right)^2\sqrt{2(2^{k-1})^2}$$
$$= \left(2^{k-1}\right)^3\sqrt{2}$$
$$= \left(2^{k-1}\right)^3\sqrt{2}. \blacksquare$$

Theorem 4.1.3 Suppose $G(\mathbb{Z}_n)$ is a unit graph of $\mathbb{Z}_n$, with $n = 2^k$ for any $k \in \mathbb{N}$. Then, the reduced sombor index of the unit graph $G(\mathbb{Z}_n)$is $\left(2^{k-1}\right)^3\sqrt{2} - \left(2^{k-1}\right)^2\sqrt{2}$.

Proof. Based on **Theorem 2.1.1** $G(\mathbb{Z}_n)$ a complete bipartite graph. Then Lestari et all found that there exist $e_1, e_2 \in E$, and all vertices in V_1 are neighbors with all vertices in $V_2$where $V_1, V_2 \subset V$. As a result, we obtain the solution based on **Definition 2.4.2** to find out The Reduced Sombor Index of $G(\mathbb{Z}_n)$ as follows.

$$SO_{red}\big(\mathrm{G}(R)\big) = \left(2^{k-1}\right)^2\sqrt{(2^{k-1}-1)^2+(2^{k-1}-1)^2}$$
$$= \left(2^{k-1}\right)^2\sqrt{2(2^{k-1}-1)^2}$$
$$= \left(2^{k-1}\right)^2(2^{k-1}-1)\sqrt{2}$$
$$= \left(\left(2^{k-1}\right)^3-\left(2^{k-1}\right)^2\right)\sqrt{2}$$
$$SO_{red}\big(\mathrm{G}(R)\big) = \left(2^{k-1}\right)^3\sqrt{2} - \left(2^{k-1}\right)^2\sqrt{2}. \blacksquare$$

Theorem 4.1.4 Suppose $G(\mathbb{Z}_n)$ is a unit graph of $\mathbb{Z}_n$, with $n = 2^k$ for any $k \in \mathbb{N}$. Then, The Average Sombor Index of the unit graph $G(\mathbb{Z}_n)$ is $\left((2^{3k-3}) - 2(2^{3k-4})\right)\sqrt{2}$.

Proof. Based on **Theorem 2.1.1** $G(\mathbb{Z}_n)$ a complete bipartite graph. Then Lestari et all found that there exist $e_1, e_2 \in E$, and all vertices in V_1 are neighbors with all vertices in V_2 where $V_1, V_2 \subset V$. As a result, the solution based on **Definition 2.4.2** to find The Average Spacing Index of $G(\mathbb{Z}_n)$is obtained as follows.

$$SO_{avg}(\mathrm{G}(R)) = \left(2^{k-1}\right)^2 \sqrt{\left(2^{k-1} - \frac{2(2^{k-1})^2}{2^k}\right)^2 + \left(2^{k-1} - \frac{2(2^{k-1})^2}{2^k}\right)^2}$$

$$= \left(2^{k-1}\right)^2 \sqrt{2\left(2^{k-1} - \frac{2\left(2^{k-1}\right)^2}{2^k}\right)^2}$$

$$= \left(2^{k-1}\right)^2 \left(2^{k-1} - \frac{2\left(2^{k-1}\right)^2}{2^k}\right)\sqrt{2}$$

$$= \left(\left(2^{k-1}\right)^3 - \frac{2\left(2^{k-1}\right)^4}{2^k}\right)\sqrt{2}$$

$$SO_{avg}(\mathrm{G}(R)) = \left((2^{3k-3}) - 2(2^{3k-4})\right)\sqrt{2}. \blacksquare$$

Theorem 4.1.5 Suppose $G(\mathbb{Z}_n)$ is a unit graph of $\mathbb{Z}_n$, with $n = 2^k$ for any $k \in \mathbb{N}$. Then, The Harmonic Index of the unit graph $G(\mathbb{Z}_n)$ is 2^{k-1}.

Proof. Based on **Theorem 2.1.1** $G(\mathbb{Z}_n)$ a complete bipartite graph. Then Lestari et all found that there exist $V_1, V_2 \subset V$, where all vertices in V_1 are neighbors to all vertices in V_2 and there are no neighboring edges in the same partition. As a result, there is only one case that needs to be solved to find out The Harmonic Index of $G(\mathbb{Z}_n)$ as follows.

$$H(\mathrm{G}(R)) = \sum_{uv \in E(G)} \frac{2}{deg(u) + deg\ (v)}$$

$$= \left(2^{k-1}\right)^2 \left(\frac{2}{2^{k-1} + 2^{k-1}}\right)$$

$$= \frac{\left(2^{k-1}\right)^2}{2^{k-1}}$$

$$H(\mathrm{G}(R)) = 2^{k-1}. \blacksquare$$

4.2 Gutman, Sombor, Reduced Sombor, Average Sombor, and Harmonics Indices on Modulo Integer Fields of order $n = p$

Teorema 4.2.1 *Suppose let $G(\mathbb{Z}_n)$ is a unit graph of $\mathbb{Z}_n$ with $n = p$, n is odd prime. Then, The Gutman Index of the unit graph $G(\mathbb{Z}_n)$* is $\frac{1}{2}(n-1)(n-2)(n^2-n)$.

Proof. Based on **Theorem 2.1.2** $G(\mathbb{Z}_n)$ a complete $\frac{n+1}{2}$ –partite graph with n odd primes. Then Lestari et all found that there exist $V_1, V_2, \ldots, V_{\frac{n+1}{2}} \subset V$ where all vertices are neighbors, and there is no edge in a partition. As a result, there are three cases to find The Gutman Index of $G(\mathbb{Z}_n)$ as follows.

Case 1. For $\{0, v\} \subseteq V$ and $v \in \mathbb{Z}_n \backslash \{0\}$, we get

$$\sum_{\{0,v\} \subseteq V} deg(u)deg(v)d(u,v) = (n-1)(n-1)(n-2)(1)$$

$$= (n-1)^2(n-2).$$

Case 2. For $u, v \in V_i$, for $i = 1,2, \dots, \frac{n+1}{2}$ and $u \neq v$, we obtain

$$\sum_{i=1}^{\frac{n-1}{2}} \sum_{\substack{u,v \in V_i \\ u \neq v}} deg(u)deg(v)d(u,v) = \frac{(n-1)}{2}(n-2)(n-2)(2)$$
$$= (n-1)(n-2)^2.$$

Case 3. For $u, v \in \mathbb{Z}_n \backslash \{0\}$, for $u \in V_i, v \in V_j, i \neq j$, we get

$$\sum_{\substack{u \in V_i \\ v \in V_j \\ i \neq j}} deg(u)deg(v)d(u,v) = \frac{(n-3)(n-1)}{2}(n-2)(n-2)(1)$$
$$= \frac{(n-3)(n-1)(n-2)^2}{2}.$$

Based on **Definition 2.4.1** and the above three cases, The Gutman Index of the unit graph $G(\mathbb{Z}_n)$ when n is an odd prime number is

$$Gut\big(\mathrm{G}(R)\big) = (n-1)^2(n-2) + \frac{(n-3)(n-1)(n-2)^2}{2} + (n-1)(n-2)^2$$
$$= (n-1)(n-2)\left((n-1) + \frac{(n-3)(n-2)}{2} + (n-2)\right)$$
$$= (n-1)(n-2)\left(\left(\frac{2n-2}{2}\right) + \frac{(n-3)(n-2)}{2} + \left(\frac{2n-4}{2}\right)\right)$$
$$= (n-1)(n-2)\left(\frac{4n-6}{2} + \frac{(n-3)(n-2)}{2}\right)$$
$$= \frac{1}{2}(n-1)(n-2)(n^2 - 5n + 6 + 4n - 6)$$
$$Gut\big(\mathrm{G}(R)\big) = \frac{1}{2}(n-1)(n-2)(n^2 - n). \blacksquare$$

Theorem 4.2.2 *Suppose $G(\mathbb{Z}_n)$ is a unit graph of $\mathbb{Z}_n$ with $n = p$, n is odd prime. Then, The Sombor Index of the unit graph $G(\mathbb{Z}_n)$ is* $(n-1)\sqrt{n^2 + (n-1)(n-5)} + \frac{1}{2}\sqrt{2}(n-3)(n-1)(n-2)$.

Proof. Based on **Theorem 2.1.2** $G(\mathbb{Z}_n)$ a complete $\frac{n+1}{2}$ −partite graph with n odd primes. Then Lestari et all found that there exist $e_1, e_2 \in E$, where all vertices are neighbors, and there is no edge inside a partition. As a result, there are two cases to find the spacing index of $G(\mathbb{Z}_n)$ as follows.

Case 1. For $\{0, v\} \subseteq V$ and $v \in \mathbb{Z}_n \backslash \{0\}$, we get

$$\sum_{\{0,v\} \subseteq V} \sqrt{deg(u)^2 + deg(v)^2} = (n-1)\sqrt{(n-1)^2 + (n-2)^2}$$
$$= (n-1)\sqrt{2n^2 - 6n + 5}.$$

Case 2. For $u, v \in \mathbb{Z}_n \backslash \{0\}$, for $u \in e_i, v \in e_j, i \neq j$, we get

$$\sum_{\substack{u\in V_j\\ v\in V_j\\ i\neq j}} \sqrt{deg(u)^2+deg(v)^2} = (n-3)\left(\frac{n-1}{2}\right)\sqrt{(n-2)^2+(n-2)^2}$$
$$= \frac{1}{2}(n-3)(n-1)\sqrt{2(n-2)^2}.$$

Based on **Definition 2.4.2** and the above two cases, The Spacing Index of the unit graph $G(\mathbb{Z}_n)$ when n is an odd prime number is.

$$SO(\text{G}(R)) = (n-1)\sqrt{2n^2-6n+5} + \frac{1}{2}(n-3)(n-1)\sqrt{2(n-2)^2}$$
$$= (n-1)\sqrt{n^2+n^2-6n+5} + \frac{1}{2}(n-3)(n-1)(n-2)\sqrt{2}$$
$$SO(\text{G}(R)) = (n-1)\sqrt{n^2+(n-1)(n-5)} + \frac{1}{2}\sqrt{2}(n-3)(n-1)(n-2).\ \blacksquare$$

Theorem 4.2.3 *Suppose $G(\mathbb{Z}_n)$ is a unit graph of $\mathbb{Z}_n$ with $n=p$, n is odd prime. Then, The Reduced Sombor Index of the unit graph $G(\mathbb{Z}_n)$* is $(n-1)\sqrt{2n^2-10n+13} + \frac{1}{2}(n-3)^2(n-1)\sqrt{2}$.

Proof. Based on **Theorem 2.1.2** $G(\mathbb{Z}_n)$ a complete $\frac{n+1}{2}$ −partite graph with n odd primes. Then Lestari et all found that there exist $e_1, e_2 \in E$, where all vertices are neighbors, and there is no edge inside a partition. As a result, there are two cases to find The Reduced Sombor Index of $G(\mathbb{Z}_n)$ as follows.

Case 1. For $\{0,v\} \subseteq V$ and $v \in \mathbb{Z}_n \backslash \{0\}$, we obtain

$$\sum_{\{0,v\}\subseteq V} \sqrt{(deg(u)-1)^2+(deg(v)-1)^2}$$
$$= (n-1)\sqrt{\left((n-1)-1\right)^2+\left((n-2)-1\right)^2}$$
$$= (n-1)\sqrt{(n-2)^2+(n-3)^2}.$$

Case 2. For $u, v \in \mathbb{Z}_n \backslash \{0\}$, for $u \in e_i, v \in e_j, i \neq j$ j, we get

$$\sum_{\substack{u\in V_j\\ v\in V_j\\ i\neq j}} \sqrt{(deg(u)-1)^2+(deg(v)-1)^2}$$
$$= (n-3)\left(\frac{n-1}{2}\right)\sqrt{\left((n-2)-1\right)^2+\left((n-2)-1\right)^2}$$
$$= \frac{1}{2}(n-3)(n-1)\sqrt{2(n-2)^2}.$$

Based on **Definition 2.4.2** and the above two cases, The Reduced Sombor Index of the unit graph $G(\mathbb{Z}_n)$ when n is an odd prime number is

$$SO_{red}(\text{G}(R)) = (n-1)\sqrt{(n-2)^2+(n-3)^2}$$
$$+\frac{1}{2}(n-3)(n-1)\sqrt{2(n-3)^2}$$

$$= (n-1)\sqrt{n^2-4n+4+n^2-6n+9}+\frac{1}{2}(n-3)^2(n-1)\sqrt{2}$$

$$SO_{red}(\mathrm{G}(R)) = (n-1)\sqrt{2n^2-10n+13}+\frac{1}{2}(n-3)^2(n-1)\sqrt{2}. \blacksquare$$

Theorem 4.2.4 *Suppose $G(\mathbb{Z}_n)$ is a unit graph of $\mathbb{Z}_n$ with $n = p$, n is odd prime. Then, The Average Sombor Index of the unit graph $G(\mathbb{Z}_n)$ is* $\frac{(n-1)}{n}\left(\sqrt{(n-1)^2+1}+\frac{(n-3)}{2}\sqrt{2}\right)$.

Proof. Based on **Theorem 2.1.2** $G(\mathbb{Z}_n)$ a complete $\frac{n+1}{2}$ −partite graph with n odd primes. Then Lestari et al. found that there exist $e_1, e_2 \in E$, where all vertices are neighbors, and there is no edge inside a partition. As a result, there are two cases to find The Average Spacing Index of $G(\mathbb{Z}_n)$ as follows.

Case 1. For $\{0, v\} \subseteq V$ and $v \in \mathbb{Z}_n \backslash \{0\}$, we obtain

$$\sum_{\{0,v\}\subseteq V}\sqrt{\left(\deg(u)-\frac{2m}{n}\right)^2+\left(\deg(v)-\frac{2m}{n}\right)^2}$$

$$= (n-1)\sqrt{\left((n-1)-\frac{\frac{2(n-1)^2}{2}}{n}\right)^2+\left((n-2)-\frac{\frac{2(n-1)^2}{2}}{n}\right)^2}$$

$$= (n-1)\sqrt{\left((n-1)-\frac{(n-1)^2}{n}\right)^2+\left((n-2)-\frac{(n-1)^2}{n}\right)^2}$$

$$= (n-1)\sqrt{\left(\frac{(n)(n-1)-(n-1)^2}{n}\right)^2+\left(\frac{(n)(n-2)-(n-1)^2}{n}\right)^2}$$

$$= (n-1)\sqrt{\frac{(n-1)^2+(1)^2}{n^2}}$$

$$= \frac{(n-1)}{n}\sqrt{(n-1)^2+1}.$$

Case 2. For $u, v \in \mathbb{Z}_n \backslash \{0\}$, for $u \in e_i, v \in e_j, i \neq j$, we get

$$\sum_{\substack{u\in V_j\\ v\in V_j\\ i\neq j}}\sqrt{\left(\deg(u)-\frac{2m}{n}\right)^2+\left(\deg(v)-\frac{2m}{n}\right)^2}$$

$$= (n-3)\left(\frac{n-1}{2}\right)\sqrt{\left((n-2)-\frac{\frac{2(n-1)^2}{2}}{n}\right)^2+\left((n-2)-\frac{\frac{2(n-1)^2}{2}}{n}\right)^2}$$

$$= \frac{(n-3)(n-1)}{2}\sqrt{\left((n-2)-\frac{(n-1)^2}{n}\right)^2+\left((n-2)-\frac{(n-1)^2}{n}\right)^2}$$
$$= \frac{(n-3)(n-1)}{2}\sqrt{2\left((n-2)-\frac{(n-1)^2}{n}\right)^2}$$
$$= \frac{(n-3)(n-1)}{2n}\sqrt{2(n^2-2n-(n^2-2n+1))^2}$$
$$= \frac{(n-3)(n-1)}{2n}\sqrt{2}.$$

Based on **Definition 2.4.2** and the above two cases, The Average Spacing Index of the unit graph $G(\mathbb{Z}_n)$ when n is an odd prime number is

$$SO_{avg}\big(\mathrm{G}(R)\big) = \frac{(n-1)}{n}\sqrt{(n-1)^2+1}+\frac{(n-3)(n-1)}{2n}\sqrt{2}$$
$$SO_{avg}\big(\mathrm{G}(R)\big) = \frac{(n-1)}{n}\left(\sqrt{(n-1)^2+1}+\frac{(n-3)}{2}\sqrt{2}\right). \blacksquare$$

Theorem 4.2.5 *Supposet $G(\mathbb{Z}_n)$ is a unit graph of $\mathbb{Z}_n$ with $n = p$, n is odd prime. Then, The Harmonic Index formed from the unit graph $G(\mathbb{Z}_n)$* is $\frac{2n-2}{2n-3}+\frac{n^2-4n+3}{2n-4}$.

Proof. Based on **Theorem 2.1.2** $G(\mathbb{Z}_n)$ a complete $\frac{n+1}{2}$−partite graph with n odd primes. Then Lestari et all found that there exist $V_1, V_2, \dots, V_{\frac{n+1}{2}} \subset V$ where all vertices are neighbors, and there is no edge in a partition. As a result, there are two cases to find The Harmonic Index of $G(\mathbb{Z}_n)$ as follows.

Case 1. For $\{0, v\} \subseteq V$ and $v \in \mathbb{Z}_n \backslash \{0\}$, we get

$$\sum_{\{0,v\}\subseteq V} \frac{2}{deg(u)+deg(v)} = (n-1)\left(\frac{2}{(n-1)+(n-2)}\right)$$
$$= \frac{2n-2}{2n-3}.$$

Case 2. For $u, v \in \mathbb{Z}_n \backslash \{0\}$, for $u \in V_i, v \in V_j, i \neq j$, we obtain

$$\sum_{\substack{u\in V_j\\ v\in V_j\\ i\neq j}} \frac{2}{deg(u)+deg(v)} = \frac{(n-3)(n-1)}{2}\left(\frac{2}{(n-2)+(n-2)}\right)$$
$$= \frac{n^2-4n+3}{2n-4}.$$

Based on **Definition 2.4.3** and the above three cases, The Harmonic Index of the unit graph $G(\mathbb{Z}_n)$ when n is an odd prime number is.

$$H\big(\mathrm{G}(R)\big) = \frac{2n-2}{2n-3}+\frac{n^2-4n+3}{2n-4}$$

$$= \frac{2n-2}{2n-3} + \frac{n^2-4n+3}{2n-4}$$
$$H(\mathrm{G}(R)) = \frac{2n-2}{2n-3} + \frac{n^2-4n+3}{2n-4}. \blacksquare$$

5. DISCUSSION

Based on the research findings presented, this study aims to formulate the general form of topological indices, namely the Gutman index, Sombor index, Reduced Sombor index, Average Sombor index, and Harmonic index. This research examines the ring of integers modulo with order $n = 2^k, k \in \mathbb{N}$, as well as $n = p$, where p is an odd prime number. The uniqueness of this study lies in its application within the context of unit graphs, distinguishing it from previous studies. To date, no research has specifically explored the use of topological indices in unit graphs, making this study a novel contribution to the analysis of graph structures based on topological indices. By formulating the general form of these indices within the framework of unit graphs, this research is expected to expand insights in graph theory and provide a foundation for further studies in discrete mathematics and combinatorics.

6. CONCLUSION

Based on the research conducted, The Gutman, Sombor, Reduced Sombor, Average Sombor, and Harmonic Indices of the unit graph in the integer algebra modulo $\mathbb{Z}_n$ with order $n = 2^k$, for a $k \in \mathbb{N}$ and $n = p$, where p is an odd prime number are obtained as follows.

a. The Gutman, Sombor, Reduced Sombor, Average Sombor Indices of the unit graph in the integer ring modulo $\mathbb{Z}_n$ with order $n = 2^k$, for a $k \in \mathbb{N}$, are
   1. The Gutman Index is $3\left(2^{k-1}\right)^4 - 2\left(2^{k-1}\right)^3$,
   2. The Sombor Index is $\left(2^{k-1}\right)^3\sqrt{2}$,
   3. The Reduced Sombor Index is $\left(2^{k-1}\right)^3\sqrt{2} - \left(2^{k-1}\right)^2\sqrt{2}$,
   4. The Average Sombor Index is $\left((2^{3k-3}) - 2(2^{3k-4})\right)\sqrt{2}$, and
   5. The Harmonic Index is 2^{k-1}.

b. The Gutman, Sombor, Reduced Sombor, Average Sombor Indices of the unit graph in the integer ring modulo $\mathbb{Z}_n$ with $n = p$, where p is an odd prime number, are
   1. The Gutman Index is $1\,\frac{1}{2}(n-1)(n-2)(n^2-n)$,
   2. The Sombor Index is $(n-1)\sqrt{n^2+(n-1)(n-5)} + \frac{1}{2}\sqrt{2}(n-3)(n-1)(n-2)$,
   3. The Reduced Sombor Index is $(n-1)\sqrt{2n^2-10n+13} + \frac{1}{2}(n-3)^2(n-1)\sqrt{2}$,
   4. The Average Sombor Index is $\frac{(n-1)}{n}\left(\sqrt{(n-1)^2+1} + \frac{(n-3)}{2}\sqrt{2}\right)$, and
   5. The Harmonic Index is $\frac{2n-2}{2n-3} + \frac{n^2-4n+3}{2n-4}$.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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