

The Non-Isolated Domination Number of a Graph

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Abstract

A subset S of the vertex set $V(G)$ of a graph G is said to be a dominating set if every vertex not in S is adjacent to at least one vertex in S . In this research, we introduce a new domination parameter called the non-isolated domination number of a graph. A subset S of V of a nontrivial graph G is said to be a non-isolated dominating set if S is a dominating set and there are no zero-degree vertices in the subgraph induced by S . The minimum cardinality taken over all non-isolated dominating sets is called the non-isolated domination number and is denoted by γ_1 . In this research, we obtained lower and upper bounds for the non-isolated domination number of a connected graph. We also determine the characterization of connected graphs that have the non-isolated domination numbers 2 and 3. Furthermore, we determine the non-isolated domination number of complete, n -partite complete, wheel, fan, star, cycle, and path graphs. We also determine the characterization of tree graphs that have the non-isolated domination number 2γ .

Keywords: domination number, isolated vertex, the non-isolated domination number

1 INTRODUCTION AND PRELIMINARIES

Let $G = (V, E)$ be a simple, undirected, and connected graph. A subset S of $V(G)$ is said to be a *dominating set* of G if every vertex not in S is adjacent to at least one vertex in S . The minimum of cardinalities of the dominating sets of G is called the *domination number* of G and is denoted by $\gamma(G)$. The set $S \subseteq V(G)$ is called the *minimum dominating set* if for every $S_0 \subseteq V(G)$ with $|S_0| < |S|$, then S_0 is not the dominating set. In 1962 the term "domination" (and its subsequent grammatical forms) was first used by O. Ore [16].

In 1979, a journal with the title The Connected Domination Number of a Graph by E. Sampathkumar and H.B. Walikar was published [17]. A dominating set S is a *connected dominating set* if it induces a connected subgraph in G . Since a dominating set must contain at least one vertex from every component of G , it follows that a connected dominating set exists for a graph G if and only if G is connected. The minimum of the cardinalities of the connected dominating sets of G is termed the *connected domination number* of G and is denoted $\gamma_c(G)$.



In this research, the author defines a new parameter in domination theory called the non-isolated dominating set. The definition of the non-isolated dominating set is motivated by the connected dominating set. We want to determine the domination number that is smaller than the connected domination number, but the dominating set does not contain an isolated vertex. A subset S of V of a graph G is said to be a *non-isolated dominating set* if S is a dominating set and there are no zero-degree vertices in the subgraph induced by S ($\langle S \rangle$) and it is denoted as a γ_1 -set. The minimum cardinality taken over all non-isolated dominating sets is called the *non-isolated domination number* and is denoted by γ_1 . A subgraph $H \subseteq G$ is called an induced subgraph if for every two vertices $u, v \in V(H)$, $uv \in E(H)$ if and only if $uv \in E(G)$. Notated as $\langle H \rangle$.

The introduction of the non-isolated dominating set provides a more balanced domination concept between flexibility and connectivity in graph structures. This idea fixes the problem with the standard dominating set, which lets dominant vertices be separate from each other. It also loosens up the rules of the connected dominating set, which says that all dominant vertices must form a fully connected subgraph. So, the non-isolated dominating set is a better and more flexible way to use graph theory in many areas, including designing communication networks, grouping data, and improving infrastructure using graph structures. This concept is significant in graph theory as it helps to understand the structure and efficiency of networks, particularly in applications such as communication, where isolated nodes can hinder functionality. Thus, determining γ_1 can provide insights into optimizing resource allocation and connectivity within the graph.

Example 1.1. Consider the following graph G

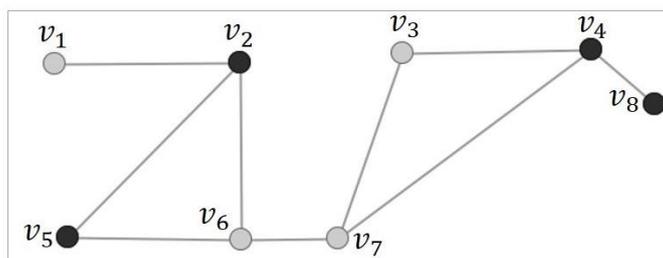


Figure 1: G with the non-isolated dominating set

In Figure 1, set $S_1 = \{v_2, v_3, v_4, v_6, v_7\}$ and $S_2 = \{v_2, v_4, v_5, v_8\}$ are the dominating sets of G . There are no isolated vertices in $\langle S_1 \rangle$ and $\langle S_2 \rangle$. Hence, S_1 and S_2 are the non-isolated dominating sets of G . Since there is no non-isolated dominating set with cardinality 3, $\gamma_1(G) = 4$.

The following are the results of previous research on domination number and connected domination number.

Theorem 2.1. (O. Ore [16]) For any graph G without isolated vertices, $\gamma(G) \leq n/2$, where n is the number of vertices in the graph G .

If G is a 4-cycle or a special kind of corona graph, it achieves the upper bound. The corona of two graphs G_1 and G_2 , as defined by Frucht and Harary, is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i th vertex of G_1 is adjacent to every vertex in the i -th copy of G_2 . The following theorem illustrates this result.

Theorem 2.2. (T.W. Haynes [9]) *If G is a graph with n vertices, where n is even, and G has no isolated vertices, then $\gamma(G) = n/2$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H .*

The following are lower and upper bounds of the connected domination number of a graph.

Theorem 2.3. (G. Mahalingam [15]) *If G is a connected graph, then*

$$\gamma(G) \leq \gamma_c(G) \leq 3\gamma(G) - 2$$

The connected domination numbers of some standard graphs can be easily found (E. Sampathkumar [14]) and are given as follows :

1. *For any complete graph of order p with $p \geq 3$, $\gamma_c(K_p) = 1$.*
2. *For any cycle graph of order p with $p \geq 4$, $\gamma_c(C_p) = p - 2$.*
3. *For any tree T of order p ,*

$$\gamma_c(T) = p - e$$

The variable e represents the number of pendant vertices in T .

4. *For any complete bipartite graph, $\gamma_c(K_{m,n}) = \begin{cases} 1 & \text{if either } m \text{ or } n = 1 \\ 2 & \text{if } m, n \geq 2 \end{cases}$*

2 MAIN RESULTS

In this paper, we obtained lower and upper bounds of the non-isolated domination number of a connected graph. We also determine the characterization of connected graphs that have the non-isolated domination numbers 2 and 3. Furthermore, we determine the non-isolated domination number of complete, n -partite complete, wheel, fan, star, cycle, and path graphs. We also determine the characterization of tree graphs that have the non-isolated domination number 2γ .

Theorem 3.1. *For any connected graph G ,*

$$2 \leq \gamma_I(G) \leq 2\gamma(G)$$

Proof. Let $S \subseteq V(G)$ be a minimum dominating set of G , so that $|S| = \gamma(G)$. By the definition of a non-isolated dominating set, we must have $\gamma_I(G) \geq 2$, since at least two vertices are required to ensure no isolated vertices in a connected graph.

For each $s_i \in S$, choose a vertex $t_i \in V(G)$ such that $s_i t_i \in E(G)$. Define the set:

$$S_0 = S \cup \{t_i \mid s_i \in S\}.$$

Since each s_i contributes at most one additional vertex t_i , we have:

$$|S_0| \leq 2\gamma(G).$$

Since S_0 forms a non-isolated dominating set, it follows that $\gamma_I(G) \leq |S_0| \leq 2\gamma(G)$. Hence, we conclude:

$$2 \leq \gamma_I(G) \leq 2\gamma(G). \blacksquare$$

The following theorem gives the characterization of connected graphs that have the non-isolated domination number 2.

Theorem 3.2. *Let G be a graph with $|V(G)| \geq 2$, $\gamma_I(G) = 2$ if and only if a star graph or double star graph is one of the spanning tree of graph G .*

Proof. (\Rightarrow) Let $\gamma_I(G) = 2$ and let $S = \{v_1, v_2\} \subseteq V(G)$ be a minimum γ_I -set. This means S is a non-isolated dominating set of minimum cardinality. If every vertex $v \in V \setminus S$ is adjacent to exactly one of v_1 or v_2 , then G has a spanning tree that is a star graph. If there exist two vertices $x, y \in V \setminus S$ such that x is adjacent to v_1 and y is adjacent to v_2 , then G has a spanning tree that is a double star graph.

(\Leftarrow) Conversely, let G has a spanning tree H that is a star graph, and let v be its center vertex.

Let the set $S = \{v, v_i\}$, where v_i is any leaf vertex in H . Since S dominates all vertices in G and contains at least two vertices, it follows that $\gamma_I(G) = 2$.

Similarly, if G has a spanning tree J that is a double star graph, let u, v be its two center vertices. Then, the set $S = \{u, v\}$ forms a non-isolated dominating set, implying $\gamma_I(G) \leq 2$. Since by definition $\gamma_I(G) \geq 2$, we conclude $\gamma_I(G) = 2$. \blacksquare

Corollary 3.1. *If G is isomorphic with a complete, n -partite complete, wheel, and fan graph, then $\gamma_I(G) = 2$.*

Proof. Let G be a graph with $|V(G)| = p$. Since there exists a vertex v such that $d(v) = p - 1$, meaning that v is a central vertex in the complete, wheel, and fan graphs, one of the spanning trees of these graphs forms a star graph. A spanning tree of an n -partite complete graph can be structured as a double star graph. By Theorem 3.2, it follows that $\gamma_I(G) = 2$. \blacksquare

Theorem 3.3. *Let $p \in \mathbb{Z}$ with $p \geq 2$ and G is isomorphic to a cycle and path graph with order p , then*

$$\gamma_I(G) = \begin{cases} \left(2 \left\lceil \frac{p}{4} \right\rceil\right) - 1 & \text{if } p \equiv 1 \pmod{4} \\ 2 \left\lceil \frac{p}{4} \right\rceil & \text{if otherwise} \end{cases}$$

Proof. The proof provided covers the path graph, while the proof for the complete graph follows the same argument.

Let $V(P_p) = \{v_i \mid i \in [1, p]\}$ and $E(P_p) = \{v_i v_{i+1} \mid i \in [1, p-1]\}$.

Case 1: $p \equiv 1 \pmod{4}$

Let m be an integer such that $p = 4m + 1$. Define the set:

$$S = \{v_{4i-2}, v_{4i-1} \mid i \in \left[1, \frac{p-1}{4}\right]\} \cup \{v_{p-1}\}$$

The cardinality of S is:

$$|S| = 2 \left(\frac{p-1}{4} \right) + 1 = 2 \left(\frac{4m+1-1}{4} \right) + 1 = 2m + 1$$

Thus,

$$\gamma_I(G) = |S| = 2 \left\lfloor \frac{p}{4} \right\rfloor - 1$$

Consider the remaining vertices:

$$V \setminus S = \{v_{4i-3}, v_{4i} \mid i \in [1, \frac{p-1}{4}]\} \setminus \{v_{p-1}\} \cup \{v_p\}$$

Since $v_{4i-3}v_{4i-2}, v_{4i}v_{4i-1}, v_{p-1}v_p \in E(G)$, the set S forms a dominating set. Moreover, the induced subgraph $\langle S \rangle$ consists of components isomorphic to K_2 and P_3 , ensuring that no isolated vertices exist in $\langle S \rangle$. Thus, S is a non-isolated dominating set, implying that $\gamma_I(G) \leq |S|$.

To prove that there is no smaller γ_I -set, assume there exists $S_0 \subseteq V(G)$ with $|S_0| = |S| - 1$. Since $|V(G)| = 4m + 1$ and $|S| = 2m + 1$, let $|S_0| = 2m$. If S_0 were a γ_I -set, then the maximum number of vertices dominated by S_0 would be at most $2m$, resulting in:

$$|S_0| + \text{dominated vertices} = 2m + 2m = 4m.$$

However, $|V(G)| = 4m + 1$, meaning there exists at least one vertex not dominated by S_0 , a contradiction. Thus, S_0 is not a γ_I -set, implying $\gamma_I(G) \geq |S|$. Since we already have $\gamma_I(G) \leq |S|$, it follows that $\gamma_I(G) = |S|$. Thus, $\gamma_I(G) = 2m + 1 = 2 \left\lfloor \frac{p}{4} \right\rfloor$, for $p \equiv 1 \pmod{4}$.

Case 2: $p \not\equiv 1 \pmod{4}$

We consider three subcases:

Subcase 1: $p = 4m$

Define:

$$S = \{v_{4i-2}, v_{4i-1} \mid i \in [1, \frac{p}{4}]\}.$$

The cardinality of S is: $|S| = 2 \frac{p}{4} = 2m$

Consider the remaining vertices:

$$V \setminus S = \{v_{4i-3}, v_{4i} \mid i \in [1, \frac{p}{4}]\}.$$

Since $v_{4i-3}v_{4i-2} \in E(G)$ and $v_{4i}v_{4i-1} \in E(G)$, each vertex in $V \setminus S$ is adjacent to a vertex in S . Furthermore, the induced subgraph $\langle S \rangle$ consists of components isomorphic to K_2 , ensuring that no isolated vertices exist in $\langle S \rangle$. Thus, S is a γ_I -set, implying that $\gamma_I(G) \leq |S|$.

Assume there exists $S_0 \subseteq V(G)$ with $|S_0| = |S| - 1$. Since $|V(G)| = 4m$ and $|S| = 2m$, let $|S_0| = 2m - 1$. Since S_0 can dominate at most $(2m - 1) + (2m - 2) = 4m - 3$ vertices, and $|V(G)| = 4m$, there exists at least one vertex not dominated by S_0 , a contradiction. Thus, $\gamma_I(G) \geq |S|$, and since $\gamma_I(G) \leq |S|$, we conclude:

$$\gamma_I(G) = 2m = 2 \left\lfloor \frac{p}{4} \right\rfloor$$

Subcase 2 : $p = 4m + 3$

Define:

$$S = \{v_{4i-2}, v_{4i} \mid i \in [1, \frac{p+1}{4}]\}$$

The cardinality of S is:

$$|S| = 2 \frac{p+1}{4} = 2m + 2$$

Consider the remaining vertices:

$$V \setminus S = \{v_{4i-3}, v_{4i-1} \mid i \in [1, \frac{p+1}{4}]\}$$

Since $v_{4i-3}v_{4i-2} \in E(G)$ and $v_{4i-1}v_{4i} \in E(G)$, each vertex in $V \setminus S$ is adjacent to a vertex in S . Moreover, the induced subgraph $\langle S \rangle$ consists of components isomorphic to K_2 , ensuring that no isolated vertices exist in $\langle S \rangle$. Thus, S is a γ_I -set, implying that $\gamma_I(G) \leq |S|$.

Assume there exists $S_0 \subseteq V(G)$ with $|S_0| = |S| - 1$. Since $|V(G)| = 4m + 3$ and $|S| = 2m + 2$, let $|S_0| = 2m + 1$. Since S_0 can dominate at most $(2m + 1) + 2m = 4m + 1$ vertices, and $|V(G)| = 4m + 3$, there exists at least one vertex not dominated by S_0 , a contradiction. Thus, $\gamma_I(G) \geq |S|$, and since $\gamma_I(G) \leq |S|$, we conclude:

$$\gamma_I(G) = 2m + 2 = 2 \left\lfloor \frac{p}{4} \right\rfloor$$

Subcase 3: $p = 4m + 2$

Define:

$$S = \{v_{4i-2}, v_{4i-1} \mid i \in [1, \frac{p+2}{4}]\} \cup \{v_{p-1}\}$$

The cardinality of S is:

$$|S| = 2 \frac{p+2}{4} = 2m + 2$$

Consider the remaining vertices:

$$V \setminus S = \{v_{4i-3}, v_{4i} \mid i \in [1, \frac{p+2}{4}]\} \setminus \{v_{p-1}\}$$

Since $v_{4i-3}v_{4i-2} \in E(G)$ and $v_{4i}v_{4i-1} \in E(G)$, each vertex in $V \setminus S$ is adjacent to a vertex in S . Moreover, the induced subgraph $\langle S \rangle$ consists of components isomorphic to K_2 , ensuring that no isolated vertices exist in $\langle S \rangle$. Thus, S is a γ_I -set, implying that $\gamma_I(G) \leq |S|$.

Assume there exists $S_0 \subseteq V(G)$ with $|S_0| = |S| - 1$. Since $|V(G)| = 4m + 2$ and $|S| = 2m + 2$, let $|S_0| = 2m + 1$. Since S_0 can dominate at most $(2m + 1) + 2m = 4m + 1$ vertices, and $|V(G)| = 4m + 2$, there exists at least one vertex not dominated by S_0 , a contradiction. Thus, $\gamma_I(G) \geq |S|$, and since $\gamma_I(G) \leq |S|$, we conclude:

$$\gamma_I(G) = 2m + 2 = 2 \left\lfloor \frac{p}{4} \right\rfloor$$

Thus, $\gamma_I(G) = 2 \left\lfloor \frac{p}{4} \right\rfloor$ for $p \not\equiv 1 \pmod{4}$. ■

For the next theorem, the definition of a triple star graph is explained as follows.

Definition 3.1. A tree graph with order p is called a triple star graph if the graph has exactly 3 vertices with more than one degree and another vertex with degree one.

Theorem 3.4. Let G be a graph with $|V(G)| \geq 5$, $\gamma_1(G) = 3$ if and only if the triple star graph is one of the spanning tree of G and G does not contain the double star graph as a spanning tree.

Proof. (\Rightarrow) Let $\gamma_1(G) = 3$, and $S = \{v_1, v_2, v_3\} \subseteq V(G)$ be a minimum γ_1 -set. Since S is a non-isolated dominating set, the induced subgraph $\langle S \rangle$ must be either a path of length 3 (P_3) or a cycle of length 3 (C_3). Since S is a dominating set, one of the spanning trees of G is a triple star graph. Since S is a minimum non-isolated dominating set, no smaller γ_1 -set $S_0 \subseteq V(G)$ exists such that $|S_0| < |S|$. Therefore, G does not contain a double star graph as a spanning tree.

(\Leftarrow) Conversely, suppose G has a spanning tree that is a triple star graph, denoted by H . For a graph of order 5, it is clear that G can be represented as P_5 , which implies $\gamma_1(G) = 3$. For a graph with more than 5 vertices, let $v_1, v_2, v_3 \in V(G)$ be the center vertices of H . Since the set $S = \{v_1, v_2, v_3\}$ is a non-isolated dominating set, it follows that $\gamma_1(G) \leq 3$.

Since G does not contain a double star graph as a spanning tree, no non-isolated dominating set $S_0 \subseteq V(G)$ of size 2 exists. Therefore, $\gamma_1(G) \geq 3$, and thus $\gamma_1(G) = 3$. ■

For the next theorem, we have determined the characterization of tree graphs that have the non-isolated domination number 2 by using the diameter of a graph.

Theorem 3.5. Let G be a tree with $|V(G)| \geq 2$, $\gamma_1(G) = 2$ if and only if $1 \leq \text{diam}(G) \leq 3$.

Proof. (\Rightarrow) Let $\gamma_1(G) = 2$, and $S = \{v_1, v_2\}$ be a minimum non-isolated dominating set of G . Since every vertex in G is adjacent to at least one of v_1 or v_2 , the maximum distance between any pair of vertices in G is at most 3. Thus, $\text{diam}(G) \leq 3$.

(\Leftarrow) If $\text{diam}(G) = 1$, then G consists of exactly two vertices, so $\gamma_1(G) = 2$. If $\text{diam}(G) = 2$, let $u, v \in V(G)$ be two vertices such that $d(u, v) = 2$. Let uwv be a path in G , where w is the intermediate vertex such that $d(u, v) = 2$. Let $L = \{u, w, v\} \subseteq V(G)$. If there exists a vertex $x \in V \setminus L$, then x must be adjacent to w , ensuring that x is fully dominated by w . Choose the set $\{u, w\}$. Since $\{u, w\}$ is a γ_1 -set, it follows that $\gamma_1 \leq 2$. By the definition of a non-isolated dominating set, we have $\gamma_1(G) \geq 2$. Thus, we conclude that $\gamma_1 = 2$.

If $\text{diam}(G) = 3$, then let $u, v \in V(G)$ be two vertices with $d(u, v) = 3$. Let uw_1w_2v be a path in G , where w_1 and w_2 are intermediate vertices such that $d(u, v) = 3$. Let $L = \{u, w_1, w_2, v\} \subseteq V(G)$. If there exists a vertex $x \in V \setminus L$, then x must be adjacent to either w_1 or w_2 , and it satisfies $|N(x)| = 1$. This means that x has exactly one neighbor within L , ensuring that it is fully dominated by w_1 or w_2 . Choose the set $\{w_1, w_2\}$. Since $\{w_1, w_2\}$ is a nonisolated dominating set, it follows that $\gamma_1 \leq 2$. By the definition of a non-isolated dominating set, we have $\gamma_1(G) \geq 2$. Thus, we conclude that $\gamma_1 = 2$. ■

The following theorem gives the characterization of tree graphs that have the non-isolated domination number 2γ . If a vertex v of G is adjacent to a pendant vertex, we refer to it as a support. Any vertex whose degree is greater than one is called an internal vertex.

Theorem 3.6. Let G be a tree graph with order $p \geq 2$, $\gamma_1(G) = 2\gamma(G)$ if and only if the following conditions hold :

- (i) Graph G has exactly γ support.
(ii) Let u and v be distinct supports, then $d(u, v) \geq 3$.

Proof. Let $n \in \mathbb{Z}$ with $n \geq 2$ and let S be a minimum dominating set of G , so that $|S| = n$.

(\Leftarrow) For each $s_i \in S$, choose $t_i \in V(G)$ such that $s_i t_i \in E(G)$ for $i \in \{1, 2, \dots, k\}$. Define $S_0 = S \cup \{t_i \mid i \in \{1, 2, \dots, k\}\}$. Since G has exactly γ support vertices, each s_i is a support. Since $d(u, v) \geq 3$ for every distinct support vertices u and v , all vertices t_i are distinct. Therefore, S_0 is a γ_I -set with $|S_0| = 2|S| = 2n$. Hence, $\gamma_I(G) \leq 2n$.

Furthermore, for any $S_0 \subseteq V(G)$ with $|S_0| = 2n - 1$, suppose that S_0 is a γ_I -set. The number of supports in G is n , then there are $|S_0| - n = n - 1$ vertices that are not supports in S_0 . Since condition (ii) holds, there is a support with zero degree in $\langle S_0 \rangle$.

This contradicts the assumption that S_0 is a γ_I -set. Hence, S_0 is not a γ_I -set, implying that $\gamma_I \geq 2n$. Therefore, $\gamma_I(G) = 2\gamma(G)$.

(\Rightarrow) Suppose $\gamma_I(G) = 2\gamma(G) = 2n$ and $I = \{v_i\}$ is the set of supports in G . Suppose that $|I| \neq n$. If $|I| > n$, then there is a support that is not an element of S . Hence, there is a pendant vertex that is not dominated by S , contradiction.

If $|I| < n$, then there exists an internal vertex in G that is not supported. Let $I_0 = \{y_i \mid i \in [1, n - |I|]\}$ be the set of internal vertices that is not supported in G .

Let v be a support in G . If $d(v_i, v) = 2$, then choose $S_I = I \cup \{t_i \mid t_i v_i \in E(G)\} \setminus \{y_i\}$. The set S_I is a γ_I -set with cardinality less than $2\gamma(G)$. Hence, $\gamma_I(G) < 2\gamma(G)$, contradiction.

Let $y_k \in V(G)$ with $y_k y_i \in E(G)$ and $d(y_k, v) = 2$. If $d(y_i, v) = 3$, then choose $S_I = I \cup \{t_i \mid t_i v_i \in E(G)\} \cup \{y_k\} \setminus \{y_i\}$. The set S_I is a γ_I -set with cardinality less than $2\gamma(G)$. Hence, $\gamma_I(G) < 2\gamma(G)$, contradiction.

Let $y_k y_j, y_l y_j \in E(G)$, and $y_k y_l \in E(G)$. If $d(y_i, y_j) = 2$ for $i \neq j$, form $S_I = I \cup \{t_i \mid t_i v_i \in E(G)\} \cup \{y_k, y_l\} \setminus \{y_i, y_j\}$. The set S_I is a γ_I -set with cardinality less than $2\gamma(G)$. Hence, $\gamma_I(G) < 2\gamma(G)$, contradiction. Therefore, G has exactly γ support vertices.

Furthermore, let S_I be a γ_I -set. Suppose that $d(u, v) \leq 2$ for distinct supports u and v . If $d(u, v) = 1$, then choose $u, v \in S$. Hence $\gamma_I < 2\gamma(G)$, contradiction.

If $d(u, v) = 2$, then choose a vertex x such that $xv, ux \in E(G)$. Define $x \in S_I$. Hence, $\gamma_I < 2\gamma$. Contradiction. Therefore, $d(u, v) \geq 3$ with u and v as distinct supports. ■

3. CONCLUSION

Let G be a nontrivial connected graph. We have established the best lower and upper bounds for the non-isolated domination number of a connected graph G , which is given by:

$$2 \leq \gamma_I(G) \leq 2\gamma(G).$$

Exact values of γ_I for some graphs

1. For every complete graph K_p with $p \geq 3$, $\gamma_I = 2$.
2. For every complete n -partite graph with $n \geq 2$, $\gamma_I = 2$.
3. For every wheel graph W_p with $p \geq 4$, $\gamma_I = 2$.
4. For every fan graph $F_{1,n}$ with $n \geq 3$, $\gamma_I(F_{1,n}) = 2$.
5. For every cycle graph C_p with $p \geq 2$ and path graph P_p with $p \geq 2$,

$$\gamma_I(G) = \begin{cases} \left(2 \left\lfloor \frac{p}{4} \right\rfloor\right) - 1 & \text{if } p = 1 \pmod{4} \\ 2 \left\lfloor \frac{p}{4} \right\rfloor & \text{if otherwise} \end{cases}$$

6. For every star graph S_p with $p \geq 4$, $\gamma_I = 2$

Tree Graph Condition

Let G be a tree graph of order $p \geq 2$ with $\gamma_I \neq 1$. The parameter $\gamma_I(G) = 2\gamma(G)$ if and only if the following conditions hold:

1. The graph G has exactly γ support vertices.
2. If u, v are distinct support vertices, then $d(u, v) \geq 3$.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this article. The authors confirmed that the paper was free of plagiarism.

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