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Numerical Simulation and Convergence of Variational Iteration Method on Kuramoto-Sivashinsky Equation

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Abstract
Mathematical physics is only one area where partial differential equations (PDEs) have shown to be an invaluable tool for problem modeling. Among the many varieties of PDEs is the Kuramoto-Sivashinsky (K-S) equation. Since analytical solutions are not always possible for partial differential equations (PDEs), numerical methods are required to find a solution. A numerical method is required to resolve this problem, just as it is for the K-S equation. The K-S issue may be solved in a number of ways, one of which is the Variational Iteration Method (VIM). The VIM is based on three main ideas: the correction function, finite variation, and the extended Lagrange multiplier. When developing the iteration formula, one may make use of the three fundamental ideas discussed before. The objective of this research is to implement the VIM numerical scheme on the K-S equation. A wave-shaped graph is obtained based on the K-S equation, which has one valley and two hills, starting with the solution at $w = 0.2$ when $x = 0$. As x increases, the solution value decreases and reaches a minimum at $w = 0.25$ when $40 \leq x \leq 50$. Subsequently, the curve ascends once more, crossing the x-axis and reaching a maximum value Even though just a small number of iterations are carried out, the Variational Iteration approach is successful in obtaining correct answers, according to the convergence analysis of the method on the K-S problem. It can be concluded that VIM is an appropriate and efficacious instrument for the resolution of equations of the K-S variety. It may be employed as a method of solution exploration and as a verification tool for the exact solution.

Keywords: Numerical Simulation, Convergence, Variational Iteration Method, Kuramoto-Sivashinsky Equation.

1. INTRODUCTION AND PRELIMINARIES

Differential equations are one of the most interesting topics in mathematics. Reason being that differential equations are a great tool for modelling a wide variety of common issues. There are two main types of differential equations: ordinary differential equations and partial differential equations. When one variable's derivatives are included in an ordinary differential equation (ODE), the equation is known as a partial differential equation (PDE) [12]. In contrast, PDEs include many independent variables. One of the problems that can be solved using PDE is the nonlinear wave problem.

Fluid mechanics, photophysics, biological diffusion processes, and chemical reaction kinetics are just a few examples of the many natural systems that exhibit nonlinear wave issues. Alternative approaches for properly solving partial differential equations (PDEs) have been developed as a consequence of the increasing study on PDE models to describe nonlinear wave situations [10]. As an example, in 1997, Ji-Huan He introduced the variational iteration method (VIM). The variational iteration method was first introduced as a row solution of nonlinear differential equations using an iteration formula [4]. The VIM relies on three primary ideas: the correction function, finite variation, and the generalized Lagrange multiplier. The variational iteration approach relies on a combination of these ideas to generate its iteration formula [15].

A number of problems have been solved using VIM, one of which is the fractional KdV-Burger-Kuramoto equation. This method finds the precise solution to the problem by using just the starting conditions [14]. An iterative strategy is used to determine the numerical solution of VIM by combining the time and spatial components. This method uses the maximum and average errors to guide iterative solution updates until convergence is reached [7]. It is possible to use VIM to solve the Kuramoto-Sivashinsky (K-S) problem and other nonlinear partial differential equations.

Several writers have investigated this classical order K-S problem using various techniques, including finite difference discretization, the Lattice Boltzmann method, the cubic B-spline finite difference collocation approach, homotopy analysis, and others [11]. Nonlinear partial differential equations of second order include, for instance, a K-S equation. Long waves in thin films, long waves between two viscous fluids [5], and unstable drift waves are some of the physical situations where the K-S equation occurs, which is a frequent nonlinear development. Plasma instability and reaction-diffusion systems [9].

The K-S equation, which describes phase turbulence, was developed by Kuramoto for reaction-diffusion systems [8]. Sivashinsky also introduced the K-S equation [13] as a model for the spread of flames on a flat surface. A common framework in the study of continuous media is the K-S equation, a collection of nonlinear PDEs. It may be used to demonstrate a variety of things, including normal and fractional differential equations, integral and integral differential equations, chaotic behavior, and the set of linear nonlinear equations [1].

The chaotic solution to the K-S problem, a partial differential equation, takes the shape of traveling waves. Over a limited geographic region, these waves do not alter their structure [6]. According to their findings, [9] proved that the K-S equation has a singular wave solution. The solitary wave solution, which is a traveling wave solution to a nonlinear PDP, has a profile that maintains its form even while moving at a constant speed and asymptotically approaches the constant when its spatial coordinates register $\pm\infty$ [10]. Many nonlinear processes make use of solitary wave solutions, which may be expressed as exponential, hyperbolic, rational, or trigonometric functions [16].

The K-S equation has been extensively studied in the fields of nonlinear dynamics and pattern formation. However, most previous research has used traditional numerical methods such as finite difference, finite element, or spectral schemes. The Variational Iteration Method (VIM) is effective for solving nonlinear differential equations, but it has not been widely applied to the K-S equation, especially regarding convergence analysis and numerical validation. Additionally, many earlier studies that employ VIM do not clearly explain the iterative process or how it converges. This study aims to address this gap by applying VIM to the K-S equation, examining its convergence, and supporting the findings with numerical simulation. In its simplest form, the general form of the K-S equation is:

$$w_t + ww_x + \alpha w_{xx} + \beta w_{xxxx} = 0, x \in \mathbb{R}, t > 0 \tag{1.1}$$

With initial conditions:

$$w(x, 0) = f(x), x \in \mathbb{R} \tag{1.2}$$

A moderately smooth function f: $\boldsymbol{Q} \to \boldsymbol{Q}$ fulfills equation (1.1) with the parameters $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x}, \boldsymbol{t})$, which is defined on $\boldsymbol{Q} \times \boldsymbol{Q}_{\boldsymbol{0}}^{+}$. In equation (1.1), $\boldsymbol{w}_{\boldsymbol{xx}}$ represents the enormous-scale instability response with $\boldsymbol{\alpha}$ being a positive constant. The damping on tiny scales, denoted as $\boldsymbol{w}_{\boldsymbol{xxxx}}$, is caused by a positive constant $\boldsymbol{\beta}$, which functions as a viscosity. By balancing the energy levels of the big and small scales, the nonlinear form $\boldsymbol{w}_{\boldsymbol{x}}$ stabilises the function and meets certain decay criteria [2].

To define the basic concept of the VIM, the following differential equation can be consider [3]:

$$\mathcal{D}w + \mathcal{Q}w + \mathcal{F}(w) = 0, \tag{1.3}$$

In this case, $\mathcal{D}$ and $\mathcal{Q}$ are bounded linear operators, meaning that $\mu_1, \mu_2 > 0$ can be found such that $\|\mathcal{D}w\| \leq \mu_1 \|w\|, \|\mathcal{Q}w\| \leq \mu_2 \|w\|$. If t is in the interval $[0, T]$, then the nonlinear operator $N(w)$ is Lipschitz continuous with $|\mathcal{F}(w) - \mathcal{F}(w)\,| \leq \mu |w - v|$, The correction function of equation (1.3) according to VIM is:

$$w_k(t) = w_{k-1}(t) + \int_0^t \lambda(\zeta)[\mathcal{D}w_{k-1} + \mathcal{Q}\widetilde{w}_{k-1} + \mathcal{F}(\widetilde{w}_{k-1})]d\zeta. \tag{1.4}$$

To determine the Lagrange multiplier λ, it can be optimally identified with partial integrals, where w_k denotes the $k-th$ iteration and $\widetilde{w}_k$ is a finite variation with the condition $\delta\widetilde{w} = 0$ to reach a stationary state. It is now possible to obtain the variational iteration formula as follows:

$$w_k(t) = w_{k-1}(t) - \int_0^t [\mathcal{D}w_{k-1} + \mathcal{Q}\widetilde{w}_{k-1} + \mathcal{F}(\widetilde{w}_{k-1})]d\zeta. \tag{1.5}$$

In order to ascertain the efficiency of VIM when applied to the K-S equation, the convergence of VIM must first be determined. The convergence of VIM is related to the initial conditions used in the iteration process. In order to guarantee the convergence of VIM, the convergence of the iteration series generated with VIM must be proven [3].

Lemma 1 [3]. Let $A: W \to V$ be a bounded linear operator and let $\{w_n\}$ be a convergent sequence in W with limit w, then $w_k \to w$ in W implies $A(w_k) \to A(w)$ in V.

Proof:

Since

$$\|Aw_k - Aw\|V = \|A(w_k - w)\|V \leq \|A\|\|w_k - w\|W, \tag{1.6}$$

hence:

$$\lim_{k\to\infty} \|Aw_k - Aw\|V \leq \|A\| \lim_{k\to\infty} \|w_k - w\|W \tag{1.7}$$

implies that

$$A(w_k) \to A(w).$$

Theorem 1 [3]: with the parameters m_2 and m given above, the nonlinear problem (1.3) has an unique solution, whenever $0 < \alpha < 1$, where $\alpha = (\mu_2 + \mu)T$.

Proof

We shall examine the situation of operator $\mathcal{D} = \frac{\partial}{\partial t}$ (without sacrificing generality) the response to this question is contingent upon the sequence of the operator $\mathcal{D}$ in equation (1.3). Consequently, the following form can be writen to represent the answer of Eq. (1.3).

$$w = f(x) - \mathcal{D}^{-1}[\mathcal{Q}w + \mathcal{F}(w)], \tag{1.8}$$

the definition of the inverse operator $\mathcal{D}^{-1}$ is given by where the function $f(x)$ is the solution to the homogeneous equation $\mathcal{D}w = 0$.

Using the equation above, we obtain , w and w^*, which are two distinct solutions to (1.3).

$$\begin{aligned} |w - w^*| &= \left| -\int_0^t [\mathcal{Q}(w - w^*) + \mathcal{F}(w) - \mathcal{F}(w^*)]dt \right| \\ &\le \int_0^t [\mathcal{Q}(w - w^*) + \mathcal{F}(w) - \mathcal{F}(w^*)]dt \\ &\le (m_2|w - w^*| + m|w - w^*|)T \\ &\le \alpha|w - w^*|, \end{aligned}$$

from which $(1 - \alpha)|w - w^*| \le 0$ is obtained. The proof is completed since $|w - w^*| = 0$ indicates that $w = w^*$ since $0 < \alpha < 1$.

We will now rewrite equation (1.5) in the operator form as follows in order to demonstrate the convergence of the variational iteration method.

$$w_k = A[w_{k-1}], \tag{1.9}$$

where the following is the form of operator A:

$$A[w] = \int_0^t [\mathcal{D}w + \mathcal{Q}w + \mathcal{F}(w)]ds. \tag{1.10}$$

Theorem 2. (Banach's fixed point theorem) [3]: Assuming the existence of a Banach space X and a nonlinear mapping $A: X \to X$, it is required to demonstrate that:

$$\|A[w] - A[v]\| \le \gamma\|w - v\|, \qquad \forall w, v \epsilon X. \tag{1.11}$$

in the event that the constant γ is such that $(\alpha + m_1 T) < 1$, it can be demonstrated that A possesses a single fixed point. In addition, the sequence (1.9) employing VIM with an arbitrary selection of $w_0 \epsilon X$, converges to the fixed point of A and

$$\|w_k - w_r\| \le \frac{\gamma^r}{1 - \gamma}\|w_1 - w_0\|. \tag{1.12}$$

Proof.

Denoting $(C[J], \|\,.\,\|)$ Banach space of all continuous functions on J with the norm defined by:

$$\|f(t)\| = \max_{t\in J}|f(t)|$$

We are going to prove that the sequence $\{w_k\}$ is a Cauchy sequence in this Banach space,

$$\begin{aligned}
\|w_k - w_r\| &= \max_{t\in J}|w_k - w_r| \\
&= \max_{t\in J}\left|-\int_0^t [\mathcal{D}(w_{k-1} - w_{r-1}) + \mathcal{Q}(w_{k-1} - u_{r-1}) + \mathcal{F}(w_{k-1}) - \mathcal{F}(w_{r-1})]ds\right| \\
&\le \max_{t\in J}\int_0^t [\mathcal{D}(w_{k-1} - w_{r-1}) + \mathcal{Q}(w_{k-1} - w_{r-1}) + \mathcal{F}(w_{k-1}) - \mathcal{F}(w_{r-1})]ds \\
&\le \max_{t\in J}\int_0^t \left[(\mu_1 + \mu_2 + \mu) + (w_{k-1} - w_{r-1})\right]ds \\
&\le \gamma\|w_{k-1} - w_{r-1}\|
\end{aligned}$$

Let, $k = r + 1$ then

$$\|w_{r+1} - w_r\| \le \gamma\|w_r - w_{r-1}\| \le \gamma^2\|w_{r-1} - u_{r-2}\| \le \cdots \le \gamma^{\mathrm{r}}\|w_1 - w_0\|$$

Derived from the triangle inequality are the subsequent equations

$$\begin{aligned}
\|w_k - w_r\| &\le \|w_{r+1} - w_r\| + \|w_{r+2} - w_{r+1}\| + \cdots + \|w_k - w_{k-1}\| \\
&\le \left[\gamma^{\mathrm{r}} + \gamma^{\mathrm{r+1}} + \cdots + \gamma^{\mathrm{k-1}}\right]\|w_1 - w_0\| \\
&\le \gamma^{\mathrm{r}}\left[1 + \gamma + \gamma^2 + \cdots + \gamma^{\mathrm{k-r-1}}\right]\|w_1 - w_0\| \\
&\le \gamma^{\mathrm{r}}\left[\frac{1-\gamma^{\mathrm{k-r}}}{1-\gamma}\right]\|w_1 - w_0\|
\end{aligned}$$

Since $0 < \gamma < 1$,so, $\left(1 - \gamma^{k-r}\right) < 1$ then:

$$\|w_k - w_r\| \le \frac{\gamma^{\mathrm{r}}}{1-\gamma}\|w_1 - w_0\|.$$

But $\|w_1 - w_0\| < \infty$ so, as $r \to \infty$ then $\|w_k - w_r\| \to 0$. We determine that $\{w_k\}$ is a Cauchy sequence in $C[J]$, hence the proof is finished and the series converges.

It is calculated that the approximate answer w_k to problem (1.3), with the highest absolute error, is:

$$\max_{t\in J}\|w_{exact} - w_k\| \le \beta, \tag{1.13}$$

where

$$\beta = \frac{\gamma^{\mathrm{r}}T\left[(\mu_1 + \mu_2)\|w_0\| + h\right]}{1-\gamma}, h = \max_{t\in J}|N(w_0)|.$$

Proof. As a consequence of Theorems 2 and (1.12), we have the following inequality:

$$\|w_k - w_r\| \leq \frac{\gamma^r}{1-\gamma}\|w_1 - w_0\|$$

as $k \to \infty$then $w_k \to w_{exact}$ and

$$\|w_1 - w_0\| = \max_{t\in J}\left|-\int_0^t [\mathcal{D}w_0 + \mathcal{Q}w_0 + \mathcal{F}(w_0)]ds\right|$$

$$\leq \max_{t\in J}\int_0^t [|\mathcal{D}w_0| + |\mathcal{Q}w_0| + |\mathcal{F}(w_0)|]ds$$

$$\leq T\left[(\mu_1 + \mu_2)\|w\| + k\right]$$

so, the maximum absolute error in the interval J is:

$$\|w_{exact} - w_k\| = \max_{t\in J}\|w_{exact} - w_k\| \leq \beta \tag{1.14}$$

This completes the proof.

2. MAIN RESULTS

Using equation (1.4), the following steps are used to make the VIM numerical scheme for the K-S equation:

First, find the K-S equation's linear and nonlinear operators:

Using equation (1.1), we can figure out the linear and nonlinear operators like this:

$\mathcal{D}w = w_t + \alpha w_{xx} + \beta w_{xxxx}$ (linear operators)

$\mathcal{Q}w = ww_x$ (nonlinear operator)

Second, construct the correction function:

To solve the K-S equation with VIM, the correction function of equation (1.1) is:

$$w_{k+1}(x,t) = w_k(x,t) + \int_0^t \lambda(\zeta)\{w_k(x,\zeta)_\zeta + \tilde{w}_k(x,\zeta)_{xx} + \tilde{w}_k(x,\zeta)_{xxxx} + \tilde{w}_k(x,\zeta)\left(\tilde{w}_k(x,\zeta)\right)_x\} d\zeta \tag{2.1}$$

Third, Determine the Lagrange multiplier:

To reach the steady state, the value of $\delta\tilde{w} = 0$, so equation (2.1) becomes:

$$\delta w_{k+1}(x,t) = \delta w_k(x,t) + \delta\int_0^t \lambda(\zeta)\left\{w_k(x,\zeta)_\zeta + w_k(x,\zeta)\left(\tilde{w}_k(x,\zeta)\right)_x + \tilde{w}_k(x,\zeta)_{xx} + k(x,\zeta)_{xxxx}\right\}d\zeta \tag{2.2}$$

By partial integral, the stationary condition is obtained, i.e:

$1 + \lambda = 0$

and

$\lambda' = 0$

So the value of the Lagrange multiplier $\lambda = -1$ is obtained. The Lagrange multiplier value obtained is then substituted. So that equation (5) becomes:

$$w_{k+1}(x,t) = w_k(x,t) - \int_0^t \left\{ w_k(x,\zeta)_\zeta + \widetilde{w}_k(x,\zeta)_{xx} + \widetilde{w}_k(x,\zeta)_{xxxx} + \widetilde{w}_k(x,\zeta)\left(\widetilde{w}_k(x,\zeta)\right)_x \right\} d \quad (2.3)$$

Fourth, iterate the correction function:
The first approximation starts with the initial conditions given by equation (2), by using the iteration formula, other components are obtained such as:

$$w(x,t) = w(x,0) = f(x)$$

For the value of $k = 0$;

$$w_1(x,t) = w_0(x,t) - \int_0^t \left\{ w_0(x,\zeta)_\zeta + \widetilde{w}_0(x,\zeta)_{xx} + \widetilde{w}_0(x,\zeta)_{xxxx} + \widetilde{w}_0(x,\zeta)\left(\widetilde{w}_0(x,\zeta)\right)_x \right\} d\zeta$$

For the value of $n = 1$;

$$w_2(x,t) = w_1(x,t) - \int_0^t \left\{ w_1(x,\zeta)_\zeta + \widetilde{w}_1(x,\zeta)_{xx} + \widetilde{w}_1(x,\zeta)_{xxxx} + \widetilde{w}_1(x,\zeta)\left(\widetilde{w}_1(x,\zeta)\right)_x \right\} d\zeta$$

For the value of $n = 2$;

$$w_3(x,t) = w_2(x,t) - \int_0^t \left\{ w_2(x,\zeta)_\zeta + \widetilde{w}_2(x,\zeta)_{xx} + \widetilde{w}_2(x,\zeta)_{xxxx} + \widetilde{w}_2(x,\zeta)\left(w_2(x,\zeta)\right)_x \right\} d\zeta$$

And so on for $k = 3, 4, \ldots$

After solving the K-S equation numerically with VIM, we will iterate with the following initial conditions and some other parameters. In this study, the Kuramoto-Sivashinsky equation was normalized so that all the parameters used were unitless (dimensionless). This nondimensionalization process uses the right length and time scales, which makes it easier to analyze and run numerical simulations. So, all of the parameters that show up in our model and simulation results don't have any physical units.

$w_t + ww_x + w_{xx} + w_{xxxx} = 0$;

With initial conditions,

$$w(x,0) = 0.1\cos\left(\frac{x}{16}\right)\left(2 + \sin\left(\frac{x}{16}\right)\right) \quad (2.4)$$

With $w_0(x,t)$ defined as the initial value:

$$w_0(x,t) = 0.1\cos\left(\frac{x}{16}\right)\left(2 + \sin\left(\frac{x}{16}\right)\right), \quad (2.5)$$

is obtained:

$$w_1(x,t) = \cos\left(\frac{x}{16}\right)\left(0.2 + 0.1\sin\left(\frac{x}{16}\right) + t\left(0.00435059\sin\left(\frac{x}{16}\right) - 0.0003125\sin\left(\frac{3x}{16}\right) - 0.001875\cos\left(\frac{x}{8}\right)0.001403198\right)\right). \quad (2.6)$$

Then, using:

$$w_1(x,\zeta) = \cos\left(\frac{x}{16}\right)\left(0.2 + 0.1\sin\left(\frac{x}{16}\right) + \zeta\left(0.00435059\sin\left(\frac{x}{16}\right) - 0.0003125\sin\left(\frac{3x}{16}\right) - 0.001875\cos\left(\frac{x}{8}\right)0.001403198\right)\right), \quad (2.7)$$

the second iteration value is obtained, namely:

$$\begin{aligned} w_2(x,t) = {} & \cos\left(\frac{x}{16}\right)\left(0.2 + 0.1\sin\left(\frac{x}{16}\right)\right) - 5.421011\times10^{-20} t \sin\left(\frac{x}{8}\right) \\ & + \cos\left(\frac{x}{16}\right)\Bigg(t\left(0.00435059\sin\left(\frac{x}{16}\right) - 0.0003125\ \sin\left(\frac{3x}{16}\right) - 0.001875\cos\left(\frac{x}{8}\right)\right. \\ & \left. + 0.001403198\right)\Bigg) + t^2\Bigg(1.23344\times10^{-5}\sin\left(\frac{x}{8}\right) - 2.60590\times10^{-5}\sin\left(\frac{x}{4}\right) \\ & + 7.324219\times10^{-7}\sin\left(\frac{3x}{8}\right) - 5.674129\times10^{-7}\cos\left(\frac{x}{16}\right) \\ & - 2.852297\times10^{-5}\cos\left(\frac{3x}{16}\right) + 6.103516\times10^{-6}\left(\frac{5x}{16}\right) - \frac{2.966166\times10^{-6}\sin\left(\frac{x}{4}\right)}{\sin\left(\frac{x}{8}\right)}\Bigg) \\ & + \frac{t^3\left(\begin{array}{c} \sin\left(\frac{x}{16}\right)\left(\begin{array}{c} 2.758209\times10^{-7}\sin\left(\frac{x}{16}\right) - 3.392599\times10^{-7}\sin\left(\frac{3x}{16}\right) \\ + 5.977948\times10^{-8}\sin\left(\frac{5x}{16}\right) - 2.034505\times10^{-9}\sin\left(\frac{7x}{16}\right) \\ - 2.711776\times10^{-7}\cos\left(\frac{x}{8}\right) + 2.352695\times10^{-7}\cos\left(\frac{x}{4}\right) \\ - 1.220703\times10^{-8}\cos\left(\frac{3x}{8}\right) + 1.137066\times10^{-7} \end{array}\right) \\ - 4.577637\times10^{-9}\sin\left(\frac{7x}{16}\right) + 1.831055\times10^{-8}\cos\left(\frac{x}{8}\right) \\ - 1.831055\times10^{-8}\cos\left(\frac{3x}{8}\right) \end{array}\right)}{\tan\frac{x}{16}} \end{aligned} \tag{2.8}$$

Then the same steps are performed for $w_3(x,t), w_4(x,t), \ldots$.

The exact solution of the K-S equation, given the initial conditions, will then be found. A single wave solution exists for the K-S equation. The traveling wave solution of a nonlinear PDP is represented by the solitary wave solution. If you solve the differential equation $w(x,t) = f(x - ct)$, you get a travelling wave solution of a partial differential equation (PDP). To implement the travelling wave solution, assumptions are used, viz:

$w(x,t) = f(\zeta)$, with $\zeta = x - ct$

Based on the given initial conditions on equation (9), this shows that at $t = 0$, $f(\xi)$ which defines the waveform, must satisfy:

$$f(x) = 0.1\cos\left(\frac{x}{16}\right)\left(2 + \sin\left(\frac{x}{16}\right)\right) \tag{2.9}$$

Thus, the form of $f(\xi)$ assumed to be the travelling wave solution at time t is:

$$w(x,t) = 0.1\cos\left(\frac{x}{16} - ct\right)\left(2 + \sin\left(\frac{x}{16} - ct\right)\right) \tag{2.10}$$

In this case c is the speed of the wave, which is assumed to be constant, i.e. $c = 1$. Thus, the exact solution is obtained, namely:

$$w(x,t) = 0.1\cos\left(\frac{x}{16} - t\right)\left(2 + \sin\left(\frac{x}{16} - t\right)\right) \tag{2.11}$$

Several simulations of the solution of the K-S equation have been performed based on the given initial conditions, namely $w(x,t)$, when $x \in [0, 32\pi], [t = 0.0004]$, and $w(x,t)$, when $x \in [0, 12.8\pi], [t = 0.0004]$. In addition, the solution with VIM and the exact solution were compared in the two conditions.

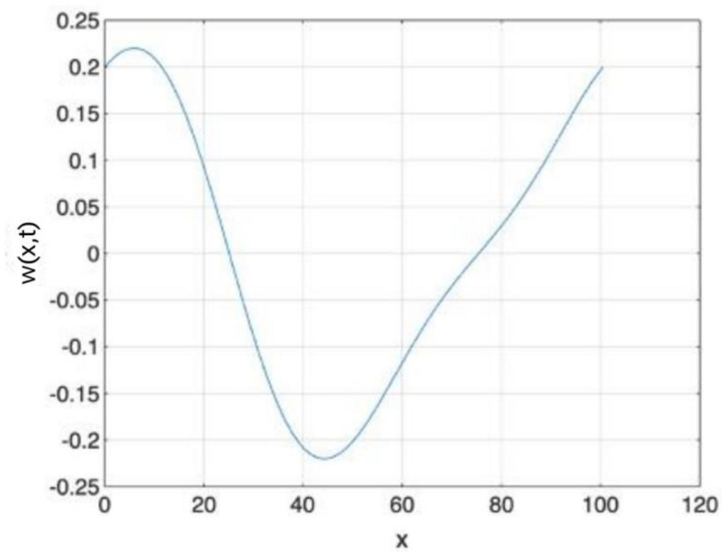

Figure 2.1 Curve showing the solution $w(x,t)$, when $x \in [0, 32\pi], [t = 0.0004]$ at the 3rd iteration of the VIM solution.

Figure 2.1 depicts the simulation result, $w(x,t)$, when $x \in [0, 32\pi], [t = 0.0004]$. The curve gives rise to a wave-like graph with one valley and two hills, beginning with the solution at the point $w \approx 0.2$ when $x = 0$. As x increases, the solution value decreases and reaches a minimum $w \approx -0.25$ when $40 \leq x \leq 50$. Subsequently, the curve ascends to a second minimum, crossing the x-axis once more. This demonstrates that the function $w(x,t)$ exhibits a sinusoidal shape, oscillating between a maximum value of approximately 0.22 and a minimum value of approximately ≈ -0.25. Consequently, the amplitude of the curve can be calculated to be 0.235.

On figure 2.1, the shape of one valley and two hills is crucial because it shows the main pattern of the waves that form inside the system. We can figure out how the solution values change at different times and places by looking at this pattern. This pattern shows how the solution changes over time, showing whether it stays stable or becomes more complicated. This form gives a first look at the system that the Kuramoto-Sivashinsky equation describes, including its features and how they work together.

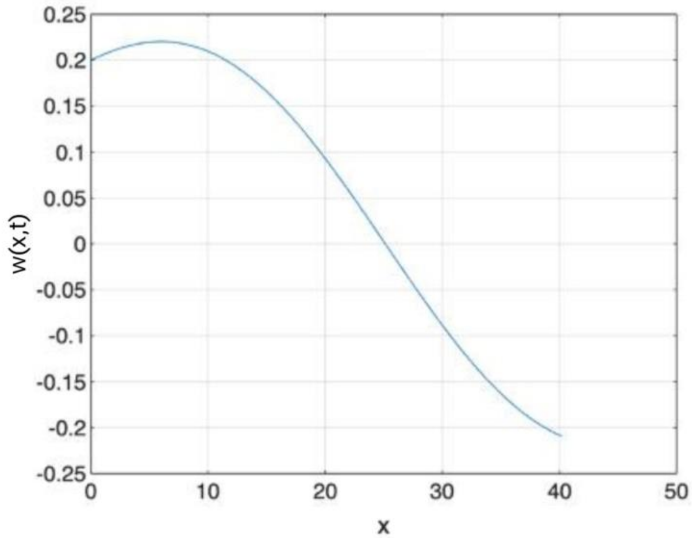

Figure 2.2 Curve representing the solution $w(x,t)$, when $x \in [0, 12.8\pi], [t = 0.0004]$ at the 3rd iteration of the VIM solution.

Figure 2.2 illustrates the simulation $w(x,t)$ when $x \in [0, 12.8\pi], [t = 0.0004]$. Based on the aforementioned curve, a curve with a single valley and hill is obtained. It can be observed that the solution begins at the point $w \approx 0.2$ when $x = 0$. In Figure 3, the function $w(x,t)$ exhibits a sinusoidal shape that oscillates between a maximum value of approximately 0.22 and a minimum value of approximately ≈ -0.22. This indicates that the amplitude of the curve is 0.22.

On figure 2.1 dan 2.2, the shape of valley and hills is crucial because it shows the main pattern of the waves that form inside the system. We can figure out how the solution values change at different times and places by looking at this curve. This curve shows how the solution changes over time, showing whether it stays stable or becomes more complicated. This form gives a first look at the system that the Kuramoto-Sivashinsky equation describes, including its features and how they work together.

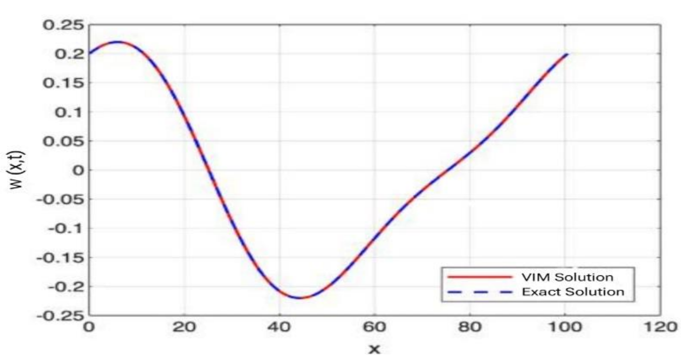

Figure 2.3 Curves representing the solution $w(x,t)$, when $x[0, 32\pi], [t = 0.0004]$ of the exact solution and the 3rd iteration of VIM.

Figure 2.3 illustrates a comparison between the VIM solution and the precise solution of the K-S equation $w(x,t)$, with values of x ranging from 0 to 32π and t ranging from 0.0004 to 999. It is clear that the VIM simulation results approach the exact solution of the K-S equation with an absolute error value of 1.58621×10^{-9}, as shown by comparing them to the precise answer. The results show that VIM is a good way to solve the K-S problem.

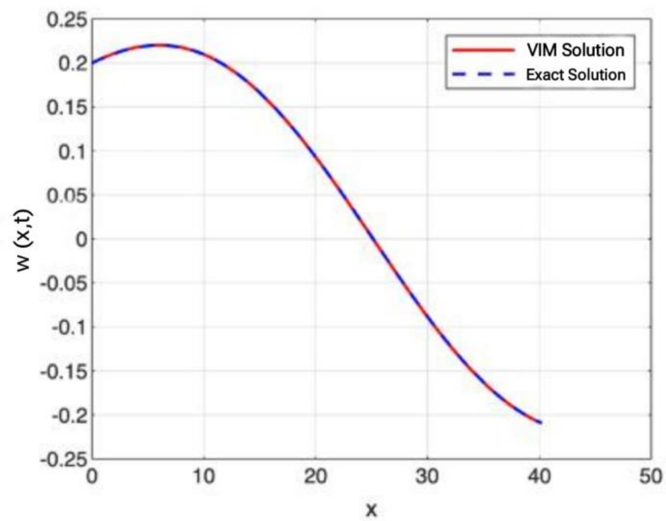

Figure 2.4 Curves representing the solution $w(x,t)$, when $x[0, 12.8\pi], [t = 0.0004]$ of the exact solution and the 3rd iteration of VIM.

The VIM solution and the actual solution of the K-S equation $w(x,t)$ are compared in Figure 2.4 for values of x from 0 to 12.8π and t =0.0004. The VIM-based solution to the K-S equation is clearly closer to the exact answer than the exact solution, having an absolute error of 3.56882×10^{-5}. The results demonstrate that VIM is an effective method for resolving the K-S issue.

The $n = 2$ strategy is used to assess the numerical solution in order to numerically test the correctness of the given methodology. This is because, as we can see from the simulations, convergence occurs after the second iteration. A mathematical application was used to acquire the value of w_n. Table 5.1 shows the comparison between w_n and w_{exact} under various scenarios.

Table 5.1 Comparison between exact and numerical solutions

$x \times \pi$	w_{exact}	w_k	Absolute Error
0	0.199959984004267	0.199999811273721	3.98273×10^{-5}
6.4	0.0913010923363756	0.0911935565907856	1.07536×10^{-4}
12.8	−0.209321534388099	−0.209357222626815	3.56882×10^{-5}
25.6	0.0323704168777061	0.0324139630591064	4.35462×10^{-5}
32	0.199959984004267	0.199999811273721	3.98273×10^{-5}

According to Table 5.1, the highest absolute error is very small and close to zero. This shows that the answer found by running the VIM numerical method twice is very accurate and very close to the exact solution. VIM has shown that it can effectively and flexibly solve difficult nonlinear equations, like the Kuramoto-Sivashinsky (K-S) problem.

3. CONCLUSION

This research presents the numerical simulation and convergence analysis of the Variational Iteration Method (VIM) used to solve this study's Kuramoto-Sivashinsky (K-S) equation. We did a numerical simulation to see how the K-S equation solutions behaved and a convergence analysis to see how well VIM worked at solving nonlinear partial differential equations. This study takes a more complete approach by looking at the convergence properties of VIM and the numerical simulations that were the primary focus of many previous studies.

The findings of the research project demonstrate that VIM has been effectively employed in the numerical solution of the K-S equation. The K-S equation is characterised by nonlinear and dissipative properties and is a widely used model for fluid dynamics and turbulent flow phenomena. From the solution curve, it can be observed that the resulting graph exhibits oscillatory behaviour, manifesting as a sinusoidal shape. Moreover, the solution depicted is a non-linear wave with constrained amplitude, which substantiates the assertion that the instability is attributable to the non-linear interaction and viscosity inherent in the K-S equation. This demonstrates that VIM is an effective method for producing accurate and efficient solutions, as evidenced by the minimal absolute error value obtained. This is particularly evident in the context of overcoming the instability caused by the non-linearity of the K-S equation.

The VIM iteration process employs the use of a Lagrange multiplier to rectify the error at each iteration step, thereby facilitating a gradual approach towards the exact solution. Furthermore, VIM offers the additional benefit of being straightforward to implement and adaptable to different initial and boundary conditions, thereby establishing it as a dependable alternative for addressing nonlinear differential equations, such as the K-S equation. Furthermore, this methodology can be extended to address other problems involving similar dynamical systems.

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