On Convergence in Norm of Functions in L^p Spaces by Convolution Using Dilation Kernel

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Abstract

In this paper, we investigate the convergence in the norm of functions in $L^p(\mathbb{R}^d)$ by convolution. We use the dilation kernel from L^1 as approximation identity and prove convergence of a function using convolution with dilation kernel in norm $\|\cdot\|_p$ for $1 \le p < \infty$ and norm $\|\cdot\|_\infty$ for $p = \infty$.

Keywords: approximation, convolution, convergence in norm, L^p spaces.

1. INTRODUCTION AND PRELIMINARIES

Convolution is a mathematical operation on two functions that leads to a function and can be considered as one modified version of these functions. Considering g and h are two functions defined on $\mathbb{R}^d$, then the convolution of g and h is defined as $(g * h)(x) = \int_{\mathbb{R}^d} g(y)h(x - y)dy$ provided that the integral converges. Convolution has applications in statistics, image processing, digital data processing, signaling, machine learning applications, fluid dynamics computing, and others [3, 4]. Many researchers have conducted a more in-depth study of convolution, such as Anastassiou [1], Gao et al. [9], Li et al. [12], Liu et al. [13], Xu & Loureiro [15], Xu & Haizhang [16], and Zhang et al. [17].

Convolution operation does not have an identity element but has an identity approximation, so the convolution of a function and an identity approximation can converge to that function. Therefore, convergence can be used to express the approximation of a function, and convolution can be used as a tool to approximate the function. Several researchers use various methods to obtain function approximations, such as Avramidou using perturbation method to modified convolution operators in $L^p(\mathbb{R})$ spaces [2], Dung et al. using translates of a single function [8], Dung & Huy modified results of Dung et al. with linear combinations [7], Bekhnazaryan using Deep Neural Networks [5], Bhaya & Abdulmunim using radial neural networks to get function approximation in L^p Spaces for $p < 1$ [6]. Moreover, Kahar explained several kernels and convolutions that can be used in approximating functions, including the Weierstrass kernel, Cauchy kernel, Jackson kernel, Gauss kernel, and Landau kernel [11]. Next, Herlinawati constructed the kernel from the sequence of functions in $L^1(\mathbb{R}^d)$ as an approximation identity [10].

This paper used the dilation kernel constructed in [10] as an identity approximation. In the case of a finite continuous function, the convergence of a finite continuous function using norm $\|\cdot\|_1$ and norm $\|\cdot\|_\infty$ [14] has been proven in previous research [10]. Now, we want to establish the convergence for function in L^p spaces. The novelty is using the dilation kernel as an approximation identity to prove convergence in norm in L^p, not only for $p = 1$ and $p = \infty$ but also for $1 \le p \le \infty$. This study aims to investigate convergence in the norm of functions in $L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, that underlie the approximation of a function by convolution. By utilizing the approximation identity from the sequence of functions in $L^1(\mathbb{R}^d)$, we want to show the convergence of $H_k * f$, $f \in L^p(\mathbb{R}^d)$, for $1 \le p \le \infty$ and $H_k \in L^1(\mathbb{R}^d)$, $k \in \mathbb{N}$. We divide it into two part. First, we proved the approximation theorem of functions in $L^p(\mathbb{R}^d)$ with $1 \le p < \infty$; in the second part, we investigated for $p = \infty$.

2. MAIN RESULTS

In this session, we discuss some theorems about the convergence in the norm of functions in $L^p(\mathbb{R}^d)$ using convolution. First, we define r-ball with centered at 0 and radius r as follows:

$$B(0, r) = \{x \in \mathbb{R}^d : \|x\| \le r\}.$$

Next, we defined the kernel on $L^1(\mathbb{R}^d)$ on Theorem 2.1.

Theorem 2.1. [10] *Let $H \in L^1(\mathbb{R}^d)$ and defined $H_k(x) = k^d H(kx)$ such that $\int_{\mathbb{R}^d} H_k(x)dx = 1$, $k \in \mathbb{N}$, and $x \in \mathbb{R}^d$. Then*

$$\int_{\mathbb{R}^d} H_k(x)dx = \int_{\mathbb{R}^d} H(x)dx \text{ and } \lim_{k \to \infty} \int_{\mathbb{R}^d \setminus B(0,r)} |H_k(x)|dx = 0$$

for $r > 0$.

In this paper, we use the kernel that satisfies Theorem 2.1. Next, suppose H_k is a sequence of functions in $L^1(\mathbb{R}^d)$ resulting from the dilation of the kernel H that satisfies Theorem 2.1 and $f \in L^p(\mathbb{R}^d)$ with $1 \le p \le \infty$, then the convergence of $H_k * f$ can be seen by observing $p < \infty$ and $p = \infty$. First, convergence of $H_k * f$ is considered for the fuction in $L^p(\mathbb{R}^d)$ with $1 \le p < \infty$ as stated in Theorem 2.2.

Theorem 2.2. *Let $H \in L^1(\mathbb{R}^d)$, $a = \int_{\mathbb{R}^d} |H(x)|dx$, $f \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$ and H_k as in the Theorem 2.1. Then*

$$\lim_{k \to \infty} \|H_k * f - af\|_p = 0.$$

Proof. Suppose $f \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$. We will prove that $\|H_k * f - af\|_p \to 0$ for $k \to \infty$. By Theorem 2.1, we obtained $\|H_k\|_1 = \int_{\mathbb{R}^d} |H_k(x)|dx = \int_{\mathbb{R}^d} |H(x)|dx = \|H\|_1$ and

$af(x) = f(x) \int_{\mathbb{R}^d} H(y)dy = \int_{\mathbb{R}^d} f(x)H_k(y)dy$. Then

$$\begin{aligned}
\|H_k * f - af\|_p^p &= \int_{\mathbb{R}^d} |(H_k * f)(x) - af(x)|^p dx \\
&= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (H_k)(y)f(x-y)dy - \int_{\mathbb{R}^d} f(x)H_k(y)dy \right|^p dx \\
&= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (H_k)(y)[f(x-y) - f(x)]dy \right|^p dx \\
&\le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(|(H_k)(y)|^{\frac{1}{p}} |(H_k)(y)|^{\frac{1}{p'}} |f(x-y) - f(x)|dy \right)^p dx
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Next, we obtained

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$$\|H_k * f - af\|_p^p \leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)|^p |(H_k)(y)| \, dy \right) \left(\int_{\mathbb{R}^d} |(H_k)(y)| \, dy \right)^{\frac{p}{p'}} dx$$

$$= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)|^p |(H_k)(y)| \, dx \right) dy$$

$$= \int_{\mathbb{R}^d} |(H_k)(y)| \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)|^p dx \right) dy$$

$$\leq \int_{B(0,r)} |(H_k)(y)| \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)|^p dx \right) +$$

$$\int_{\mathbb{R}^d \setminus B(0,r)} |(H_k)(y)| \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)|^p dx \right) dy.$$

Let $\epsilon > 0$. Since $C_0(\mathbb{R}^d)$, a set of functions with compact support, dense in $L^p(\mathbb{R}^d)$, then select function $g \in C_0(\mathbb{R}^d)$ that satisfies $\int_{\mathbb{R}^d} |f(x) - g(x)|^p dx < \epsilon$. Note that g is a function with compact support, then there exists $r > 0$ such that for $y \in B(0, r)$ satisfies

$$\int_{\mathbb{R}^d} |g(x-y) - g(x)|^p dx < \epsilon.$$

Note that for $a, b \in \mathbb{R}$ and $p > 1$, we have

$$|a + b|^p \leq (|a| + |b|)^p$$
$$= \left(\frac{1}{2}(2|a| + 2|b|) \right)^p$$
$$\leq \frac{1}{2}((2|a|)^p + (2|b|)^p)$$
$$= 2^{p-1}(|a|^p + |b|^p). \tag{2.1}$$

Next, suppose $c = \|H\|_1^{\frac{p}{p'}} 2^p \|f\|_p^p$, by (2.1), then for $y \in B(0, r)$,

$$\int_{\mathbb{R}^d} |f(x-y) - f(x)|^p dx \leq \int_{\mathbb{R}^d} |f(x-y) - g(x-y) + g(x-y) - g(x) + g(x) - f(x)|^p dx$$

$$\leq 2^{p-1} \int_{\mathbb{R}^d} |f(x-y) - g(x-y)|^p dx + \int_{\mathbb{R}^d} |g(x-y) - g(x)|^p dx + \int_{\mathbb{R}^d} |g(x) - f(x)|^p dx$$

$$\leq \frac{c}{2\|H\|_1^{\frac{p}{p'}} \|f\|_p^p} \left(\int_{\mathbb{R}^d} |f(x-y) - g(x-y)|^p dx + \int_{\mathbb{R}^d} |g(x-y) - g(x)|^p dx + \int_{\mathbb{R}^d} |g(x) -$$

$$f(x)|^p dx \right) \leq \frac{c}{2\|H\|_1^{\frac{p}{p'}} \|f\|_p^p} \epsilon. \tag{2.2}$$

Therefore, by (2.2), we have

$$\|H\|_1^{\frac{p}{p'}} \int_{B(0,r)} |(H_k)(y)| \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)|^p dx \right) \leq \|H\|_1^{\frac{p}{p'}} \frac{c}{2\|H\|_1^{\frac{p}{p'}} \|f\|_p^p} \epsilon \leq \frac{c}{2} \epsilon. \tag{2.3}$$

Next, by (2.1), for $f \in L^p$,

$$\int_{\mathbb{R}^d} |f(x-y) - f(x)|^p dx \leq 2^{p-1} \int_{\mathbb{R}^d} |f(x-y)|^p dx + 2^{p-1} \int_{\mathbb{R}^d} |f(x)|^p dx$$
$$\leq 2^p \|f\|_p^p. \tag{2.4}$$

By Theorem 2.1, there exist $r > 0$, then there is $k' > 0$ such that for every $k > k'$ satisfies

$$\int_{\mathbb{R}^d \setminus B(0,r)} |(H_k)(y)| \, dy < \frac{\epsilon}{2}. \tag{2.5}$$

By (2.4) and (2.5), obtained

$$\|H\|_1^{\frac{p}{p'}} \int_{\mathbb{R}^d \setminus B(0,r)} |(H_k)(y)| \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)|^p dx \right) dy \leq c \int_{\mathbb{R}^d \setminus B(0,r)} |(H_k)(y)| \, dy < \frac{c}{2} \epsilon. \tag{2.6}$$

By (2.3) and (2.6), we obtain $\|H_k * f - af\|_p^p \leq \frac{c}{2}\epsilon + \frac{c}{2}\epsilon = c\epsilon \to 0$ with c not depending on ϵ and $\|H_k * f - af\|_p^p \to 0$ for $k \to \infty$. As a result, $\|H_k * f - af\|_p \to 0$ untuk $k \to \infty$. ∎

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Theorem 2.2 said that $H_k * f$ is converge in norm-p to af for $p < \infty$. For $p = \infty$, consider the following theorem.

Theorem 2.3. *Let H_k satisfy Theorem 2.1, and $a = \int_{R^d} |H(x)| dx$. If $f \in L^\infty (\mathbb{R}^d)$ and is continuous in the open subset U of $\mathbb{R}^d$, then*

$$H_k * f \to af$$

and the convergence is uniform on the compact subset from U for $k \to \infty$.

Proof. Let $f \in L^\infty(\mathbb{R}^d)$. We will show that $H_k * f \to af$ for $k \to \infty$. If $z = ky$, $z \in \mathbb{R}^d$, then

$H_k * f(x) = \int_{\mathbb{R}^d} f(x - y) H_k(y) dy = \int_{\mathbb{R}^d} k^d f(x - y) H(ky) dy = \int_{\mathbb{R}^d} f(x - y) H(z) dz.$

Next, let $B(0, R)$ be a ball di $\mathbb{R}^d$ centered at zero and radius R. Suppose that K is a compact subset of U. Then

$\sup_{x \in K} |H_k * f(x) - af(x)|$

$\leq \sup_{x \in K} \left(\left| \int_{B(0,R)} [f(x - tz) - f(x)] H(z) dz \right| + \left| \int_{\mathbb{R}^d \setminus B(0,R)} [f(x - tz) - f(x)] H(z) dz \right| \right)$

$\leq \int_{B(0,R)} H(z) dz \sup_{x \in K, z \in B(0,R)} |f(x - tz) - f(x)| + 2\|f\|_\infty \int_{\mathbb{R}^d \setminus B(0,R)} |H(z)| dz$

$\leq \|H\|_1 \sup_{x \in K, z \in B(0,R)} |f(x - tz) - f(x)| + 2\|f\|_\infty \int_{\mathbb{R}^d \setminus B(0,R)} |H(z)| dz$

because f continues, then f is uniformly continuous on a compact subset of U such that

$\limsup_{k \to \infty} \left(\sup_{x \in K} |H_k * f(x) - af(x)| \right) \leq 2\|f\|_\infty \int_{|z| > R} |H(z)| dz \to 0$ for $R \to \infty$. ∎

Theorem 2.3 shows that $H_k * f$ converge in norm in L^p to af for $p = \infty$. In this case, the convergence of $H_k * f$ in the norm $\|.\|_\infty$ is uniform on subsets compact in $\mathbb{R}^d$.

3. CONCLUSION

Convergence in norm-p of function in $L^p(\mathbb{R}^d)$ with $1 \leq p \leq \infty$ has been proven by utilizing the dilation of the kernel from the sequence of functions in $L^1(\mathbb{R}^d)$ as an approximation identity. For $p = \infty$, the convergence is uniform in the compact subset in $\mathbb{R}^d$. In this research, approximation identities are constructed by the dilation of a function. For further research, readers can use other functions to construct approximation identities and study the convergence of functions in generalizations of Lebesgue spaces.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest

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