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Topological Index of Coprima Graph of Integers Group Modulo with Order of Prime Power

Indeks Topologi Graf Coprima dari Grup Bilangan Bulat Modulo dengan Orde Pangkat Prima

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Abstract
This paper explores the topological indices of the coprime graph defined on the group of integers modulo a prime power. A coprime graph is constructed with vertices representing group elements, where two vertices are adjacent if and only if their orders are relatively prime. The study focuses on analyzing four topological indices: the ABC index, Forgotten index, Gourava index, and Nirmala index. This work aims to enhance the understanding of the topological properties of coprime graphs within the structure of integer modulo groups of prime power order.

Keywords: ABC Index, Forgotten Index, Gourava Index, Nirmala Index, Coprime Graph.

Abstrak
Penelitian ini membahas indeks topologi dari graf koprima pada grup bilangan bulat modulo dengan ordo pangkat prima. Graf koprima didefinisikan sebagai graf dengan simpul berupa elemen grup, di mana dua simpul dihubungkan jika orde keduanya relatif prima. Penelitian ini difokuskan pada analisis beberapa indeks topologi, yaitu indeks ABC, indeks Forgotten, indeks Gourava, dan indeks Nirmala. Studi ini bertujuan untuk memperluas pemahaman mengenai karakteristik topologi graf koprima dalam konteks struktur grup bilangan bulat modulo.

Kata kunci: Indeks ABC, Indeks Forgotten, Indeks Gourava, Indeks Nirmala, Graf Koprima

1. INTRODUCTION AND PRELIMINARIES

The concept of topological indices was first introduced by Harry Wiener in 1947, where he developed the Wiener index to analyze the relationship between molecular structures and their physical properties [15]. In graph theory, topological indices are utilized to quantify various properties of a graph, such as node connectivity, path lengths, and structural relationships among elements in a system [4].

Graph theory is a field within discrete mathematics focused on analyzing the characteristics and uses of mathematical models that involve nodes and connections between them. Modern graph theory finds its roots in Euler's 1736 exploration of the Königsberg Bridge Problem, which set the stage for further developments in the field [5]. In algebra, graph theory has been widely employed to represent and explore structures such as groups and rings.

As graph theory developed, several graph constructions related to group theory emerged, including the coprime graph, first introduced by Ma et al. In a coprime graph constructed from a group, two different elements are joined by an edge when the greatest common divisor of their orders equals one [11]. This study investigates the topological indices of coprime graphs constructed over the group of integers modulo a prime power.

2. Fundamental Terminology

This section introduces the topological indices analyzed in this research, namely the ABC index, Forgotten index, Gourava index, and Nirmala index, which are defined on the coprime graph of the integer group modulo α, where α is a prime power. The definitions used throughout this study are as follows:

Definition 2.1. [8] *The coprime graph is a graph with nodes that are elements of a group, and two distinct nodes a and b are directly connected if and only if* $(|\mathrm{a}|, |\mathrm{b}|) = 1$.

To clarify the group structure discussed in this paper, we provide the definition of the set of integers modulo α below:

Definition 2.2. [3] *Integers modulo n form a finite set* $\{0,1,2 \dots, \alpha - 1\}$ *under addition modulo* α, *and this structure is denoted by* $\mathbb{Z}_{\varphi}$.

We extend this understanding with a definition related to the identity element in a group of prime power order:

Definition 2.3. [11] *Let G be a group and let $x \in G$, such that x is a power of a natural number and $x^k = e$, where e is the identity element. Then the order of x is $|x| = k$.*

To support the subsequent analysis, we first recall some basic graph-theoretic concepts, including simple graphs, vertex degree, paths, and bipartite graphs.

Definition 2.4. [1] *A simple graph is an undirected graph that contains neither self-loops nor multi-edges, meaning that each pair of distinct vertices is connected by at most one edge, and no edge connects a vertex to itself.*

Definition 2.5. [1] *The degree of a vertex y, denoted $deg(v)$ is the number of edges incident to v, with each self-loop (if any) counted twice.*

Definition 2.6. [1] *A path from vertex v_0 to v_n is an alternating sequence of vertices and edges*

$$\langle v_0, e_1, v_1, e_2, \dots \dots \dots, e_n, v_n \rangle$$

Such that each edge e_i connects v_{i-1} to v_i, and no vertex or edge appears more than once.

Definition 2.7. [6] *A bipartite graph is an undirected graph whose vertex set can be partitioned into two disjoint subsets, u and v, such that each edge in the graph connects a vertex from u to a vertex from v, and no edge exists between any pair of vertices within the same subset.*

In graph theory, a simple graph is an undirected graph with no loops or multiple edges. A path is a sequence of distinct vertices where each consecutive pair is connected by an edge. A graph is bipartite if its vertex set can be divided into two disjoint sets such that every edge connects a vertex from one set to the other. The degree of a vertex v, denoted $\deg(v)$,is the number of edges incident to v.

Two fundamental graph concepts used in analyzing the coprime graph are the degree of a vertex and the distance between vertices:

Definition 2.8. [13] *The vertex's degree refers to how many edges are directly connected to it within the graph.*

Definition 2.9. [13] *The distance in a graph G referes to the minimum number of edges forming a path that connects any two distinct vertices.*

Since the coprime graph on $\mathbb{Z}_\varphi$ for $\varphi = p^k$ forms a complete bipartite graph, we include the following definition:

Definition 2.10. [12] *The complete bipartite graph is one where vertices are split into two distinct groups, and each vertex from one group connects to all vertices in the other group, with no edges among vertices within the same group.*

The topological indices used in this study are defined as follows:

Definition 2.11. [2] *The Atom-bond Connectivity (ABC) index of a graph, written as ABC(G), quantifies the contributions of every connected vertex pair s and t, and is formulated as:*

$$ABC(G) = \sum_{st \in E(G)} \sqrt{\frac{deg(s) + deg(t) - 2}{deg(s)deg(t)}}$$

where $deg(s)$ and $deg(t)$ denote the degrees of vertices s and t, respectively.

Definition 2.12. [7] *Let G be a graph with vertex set $V(G)$. The forgotten index, denoted as $F(G)$, is define as the sum of the cubes of degrees of all vertices in the graph:*

$$F(G) = \sum_{s \in V(G)} deg(s)^3$$

Where $deg(s)$ represents the degree of vertex s in G.

Definition 2.13. [10] *Let $G = (V, E)$, be a simple connected graph, for any vertex $s \in V$, the degree $deg(s)$ refers to the number of vertices that are directly connected to it. The first Gourava Index, denoted as $GO_1(G)$, is determined by summing, over all edges $st \in E$, the value:*

$$GO_1(G) = \sum_{st \in E(G)} [(deg(s) + deg(t)) + deg(s)deg(t)]$$

Definition 2.14. [14] *The Nirmala index of a graph G, denoted as $N(G)$, is determined by:*

$$N(G) = \sum_{st \in E(G)} \sqrt{deg(s) + deg(t)}$$

Where $deg(s)$ and $deg(t)$ are the degrees of adjacent vertices s and t.

3. MAIN RESULTS

To analyze the topological indices, it is essential to first examine the structure of the coprime graph under consideration. The lemma below establishes a key structural property of the coprime graph formed over the integers modulo a power of a prime number, which serves as the basis for computing the indices.

Lemma 3.1. [9] *Let φ can be expressed as p^k, where p is prime and k is natural number. Then, the coprime graph defined on $\mathbb{Z}_\varphi$ forms a complete bipartite structure.*

Having established the basic structure of the coprime graph, we proceed to calculate each of the topological indices beginning with the ABC index.

Theorem 3.2. *Let $\Gamma_{\mathbb{Z}_\varphi}$ be the coprime graph associated with the integer group modulo φ, where $\varphi = p^k$, with p prime and $k \in \mathbb{N}$. Then,*

$$ABC\left(\Gamma_{\mathbb{Z}_\varphi}\right) = (\varphi - 1)\sqrt{\frac{(\varphi - 2)}{(\varphi - 1)}}$$

Proof. *Let $\Gamma_{\mathbb{Z}_\varphi}$ represent the coprime graph constructed from the group $\mathbb{Z}_\varphi$. Based on **Lemma 3.1**, this graph exhibits the structure of a complete bipartite graph. Consequently, the identity element in the group has a degree $\varphi - 1$, while all other vertices have degree 1. Hence, the ABC index of $\Gamma_{\mathbb{Z}_\varphi}$ is expressed as:*

$$ABC\left(\Gamma_{\mathbb{Z}_\varphi}\right) = \sum_{st \in E(G)} \sqrt{\frac{deg(s) + deg(t) - 2}{deg(s)deg(t)}}$$

$$= \sqrt{\frac{deg(0) + deg(1) - 2}{deg(0)deg(1)}} + \sqrt{\frac{deg(0) + deg(2) - 2}{deg(0)deg(2)}} + \cdots \ldots \ldots$$

$$+ \sqrt{\frac{deg(0) + deg(\varphi - 1) - 2}{deg(0)deg(\varphi - 1)}}$$

$$= \sqrt{\frac{(\varphi - 1) + (1) - 2}{(\varphi - 1)(1)}} + \sqrt{\frac{(\varphi - 1) + (1) - 2}{(\varphi - 1)(1)}} + \cdots \ldots \ldots$$

$$+ \sqrt{\frac{(\varphi - 1) + (1) - 2}{(\varphi - 1)(1)}}$$

$$= (\varphi - 1)\sqrt{\frac{(\varphi - 2)}{(\varphi - 1)}}$$

Therefore, we conclude that the ABC index of the coprime graph on $\mathbb{Z}_\varphi$ where $\varphi = p^k$ is given by $(\varphi - 1)\sqrt{\frac{(\varphi-2)}{(\varphi-1)}}$ ■

Example 3.1. *Given $\Gamma_{\mathbb{Z}_3}$, the coprime graph of $\mathbb{Z}_3 = \{0,1,2\}$. According to **Lemma 3.1**, this graph forms a complete bipartite structure. The identity element (0) has degree 2, while the other vertices each have degree 1. The ABC index of $\mathbb{Z}_3$ can be calculated as follows:*

$$ABC(G) = \sum_{st \in E(G)} \sqrt{\frac{deg(s) + deg(t) - 2}{deg(s)deg(t)}}$$
$$= \sqrt{\frac{deg(0) + deg(1) - 2}{deg(0)deg(1)}} + \sqrt{\frac{deg(0) + deg(2) - 2}{deg(0)deg(2)}}$$
$$= \sqrt{\frac{(2) + (1) - 2}{(2)(1)}} + \sqrt{\frac{(2) + (1) - 2}{(2)(1)}}$$
$$= \sqrt{\frac{(2) + (-1)}{2}} + \sqrt{\frac{(2) + (-1)}{2}}$$
$$= 2\sqrt{\frac{1}{2}}$$

We now proceed to compute the Forgotten index, as outlined in the upcoming theorem.

Theorem 3.3. *Let $\Gamma_{\mathbb{Z}_\varphi}$ be the coprime graph of the group $\mathbb{Z}_\varphi$, where $\varphi = p^k$ with p is a prime number and $k \in \mathbb{N}$. Then,*

$$F\left(\Gamma_{\mathbb{Z}_\varphi}\right) = (\varphi - 1)^3 + (\varphi - 1)$$

Proof. *Let $\Gamma_{\mathbb{Z}_\varphi}$ be the coprime corresponding to the group $\mathbb{Z}_\varphi$. Referring to **Lemma 3.1**, it is known that this graph is structured as a complete bipartite graph. In this configuration, the identity element has a degree of $\varphi - 1$, while each remaining vertex has degree 1. Hence, the forgotten index for $\Gamma_{\mathbb{Z}_\varphi}$ can be evaluated as:*

$$F = \sum_{s \in V(G)} deg(s)^3$$
$$= deg(0)^3 + deg(1)^3 + deg(2)^3 + \cdots \ldots \ldots + deg(\varphi - 1)$$
$$= (\varphi - 1)^3 + (1)^3 + (1)^3 + \cdots \ldots \ldots + (1)^3$$
$$= (\varphi - 1)^3 + (\varphi - 1)$$

Therefore, the forgotten index for the coprime graph over the group of integers modulo $\varphi = p^k$ is confirmed to be $(\varphi - 1)^3 + (\varphi - 1)$ ∎

Example 3.2. *Given $\Gamma_{\mathbb{Z}_5}$, the coprime graph associated with the group $\mathbb{Z}_5 = \{0,1,2,3,4\}$. Based on **Lemma 3.1**, the graph exhibits a complete bipartite structure. The identity vertex has degree 4, while the remaining vertices each have degree 1. The Forgotten index of $\mathbb{Z}_5$ is calculated as:*

$$F(G) = \sum_{s \in V(G)} deg(s)^3$$
$$= deg(0)^3 + deg(1)^3 + deg(2)^3 + deg(3)^3 + deg(4)^3$$
$$= (4)^3 + (1)^3 + (1)^3 + (1)^3 + (1)^3$$
$$= (4)^3 + 3$$
$$= 67$$

We now proceed to the computation of the first Gourava index.

Theorem 3.4. *Let $\Gamma_{\mathbb{Z}_\varphi}$ denote the coprime graph of the group $\mathbb{Z}_\varphi$, with $\varphi = p^k$, where p is a prime number and $k \in \mathbb{N}$. Then the first Gourava index is given by:*

$$GO_1(\Gamma_{\mathbb{Z}_\varphi}) = (\varphi - 1)(2\varphi - 1)$$

Proof. *Let $\Gamma_{\mathbb{Z}_\varphi}$ be the coprime graph constructed from the group $\mathbb{Z}_\varphi$, where $\varphi = p^k$. Based on **Lemma 3.1**, this graph forms a complete bipartite graph. The vertex representing the identity element has degree $\varphi - 1$ and all other vertices are of degree 1. Therefore, the first Gourava index of $\Gamma_{\mathbb{Z}_\varphi}$ can be expressed as:*

$$
\begin{aligned}
GO_1(G) &= \sum_{st\in E(G)} [(deg(s) + deg(t)) + deg(s)deg(t)] \\
&= [(deg(0) + deg(1)) + deg(0)deg(1)] \\
&\quad + [(deg(0) + deg(2)) + deg(0)deg(2)] + \cdots \dots \dots \\
&\quad + [(deg(0) + deg(\varphi - 1)) + deg(0)deg(\varphi - 1)] \\
&= [((\alpha - 1) + (1)) + (\alpha - 1)(1)] + [((\alpha - 1) + (1)) + (\alpha - 1)(1)] + \cdots \dots \dots \\
&\quad + [((\varphi - 1) + (1)) + (\varphi - 1)(1)] \\
&= [(\varphi) + (\varphi - 1)] + [(\varphi) + (\varphi - 1)] + \cdots \dots \dots + [(\varphi) + (\varphi - 1)] \\
&= \varphi[\varphi + (\varphi - 1)] \\
&= (\varphi - 1)(2\varphi - 1)
\end{aligned}
$$

In conclusion, the first Gourava index for the coprime graph over the integers modulo $\varphi = p^k$ is confirmed as $(\varphi - 1)(2\varphi - 1)$ ■

Example 3.3. *Given $\Gamma_{\mathbb{Z}_3}$, representing the coprime graph of the modular group $\mathbb{Z}_3 = \{0,1,2\}$. Based on **Lemma 3.1**, this graph is also a complete bipartite graph. The identity element has degree 2, and the rest have degree 1. The Forgotten index of $\mathbb{Z}_3$ is determined as:*

$$
\begin{aligned}
GO_1(G) &= \sum_{st\in E(G)} [(deg(s) + deg(t)) + deg(s)deg(t)] \\
&= [(deg(0) + deg(1)) + deg(0)deg(1)] + [(deg(0) + deg(2)) \\
&\quad + deg(0)deg(2)] \\
&= [((2) + (1)) + (2)(1)] + [((2) + (1)) + (2)(1)] \\
&= 5 + 5 \\
&= 10
\end{aligned}
$$

Finally, we compute the Nirmala index, as follows:

Theorem 3.5. *Let $\Gamma_{\mathbb{Z}_\varphi}$ be the coprime graph* defined over the group *$\mathbb{Z}_\varphi$, where $\varphi = p^k$ with p a prime and $k \in \mathbb{N}$. Then the Nirmala index is given by:*

$$N(\Gamma_\varphi) = (\varphi - 1)\sqrt{\varphi}$$

Proof. *Let $\Gamma_{\mathbb{Z}_\varphi}$ denote the coprime graph of $\mathbb{Z}_\varphi$, where $\varphi = p^k$. According to **Lemma 3.1**, the graph has a complete bipartite form. In this structure, the identity vertex has degree $\varphi - 1$, and all others have degree 1. The Nirmala index of $\Gamma_{\mathbb{Z}_\varphi}$ is then calculated as:*

$$
\begin{aligned}
N(G) &= \sum_{st\in E(G)} \sqrt{deg(s) + deg(t)} \\
&= \sqrt{deg(0) + deg(1)} + \sqrt{deg(0) + deg(2)} + \cdots \dots \dots \\
&\quad + \sqrt{deg(0) + deg(\varphi - 1)} \\
&= \sqrt{(\varphi - 1) + (1)} + \sqrt{(\varphi - 1) + (1)} + \cdots \dots \dots + \sqrt{(\varphi - 1) + (1)} \\
&= (\varphi - 1)\sqrt{(\varphi - 1) + (1)} \\
&= (\varphi - 1)\sqrt{\varphi}
\end{aligned}
$$

Therefore, it is confirmed that the Nirmala index for the coprime graph over the group of integers modulo $\varphi = p^k$ equals $(\varphi - 1)\sqrt{\varphi}$ ■

Example 3.4. *Given $\Gamma_{\mathbb{Z}_5}$, the coprime graph derived from $\mathbb{Z}_5 = \{0,1,2,3,4\}$. As noted in **Lemma 3.1**, this graph is complete bipartite. The identity vertex has degree 4, while the remaining ones each have degree 1. The Nirmala index is computed as:*

$$\begin{aligned} N(G) &= \sum_{st\in E(G)} \sqrt{deg(s)+deg(t)} \\ &= \sqrt{deg(0)+deg(1)} + \sqrt{deg(0)+deg(2)} + \sqrt{deg(0)+deg(3)} \\ &\qquad + \sqrt{deg(0)+deg(4)} \\ &= \sqrt{(4)+(1)} + \sqrt{(4)+(1)} + \sqrt{(4)+(1)} + \sqrt{(4)+(1)} \\ &= 4\sqrt{5} \end{aligned}$$

4. CONCLUSION

From the findings of this research, the values for the four topological indices on the coprime graph constructed over the group of integers modulo a prime power are summarized as follows:

$$ABC\left(\Gamma_{\mathbb{Z}_\varphi}\right) = (\varphi-1)\sqrt{\frac{(\varphi-2)}{(\varphi-1)}},$$

$$F\left(\Gamma_{\mathbb{Z}_\varphi}\right) = (\varphi-1)^3 + (\varphi-1),$$

$$GO_1(\Gamma_{\mathbb{Z}_\varphi}) = (\varphi-1)(2\varphi-1),$$

$$N\left(\Gamma_\varphi\right) = (\varphi-1)\sqrt{\varphi},$$

CONFLICT OF INTEREST

The authors state that the research was carried out independently, without any potential conflict of interest.

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