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Connectivity Indices of Coprime Graphs over Generalized Quaternion Groups of a Certain Order

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Abstract
The generalized quaternion group is a non-abelian group of order $4n$ that exhibits certain structural similarities with the dihedral group. It is generated by two elements that satisfy specific defining relations. Meanwhile, a coprime graph is constructed by representing the elements of a group as vertices, where two vertices are adjacent if the orders of the corresponding elements are coprime. In this study, we investigate coprime graphs derived from generalized quaternion groups, particularly when the group order is given by $n = 2p$, with p being a prime number. Based on the structural properties of these graphs, we compute several connectivity indices, including the First and Second Zagreb indices, the Wiener index, the hyper-Wiener index, the Harary index, and the Szeged index.
Keywords: Connectivity Index, Coprime Graph, Generalized Quaternion Group

1. INTRODUCTION AND PRELIMINARIES

Representing groups using graphs is an effective approach to better understand their structure. This idea was pioneered by Artur Cayley in 1843, leading to what is now known as Cayley graphs. Subsequent developments extended this concept by replacing groups with semigroups, as explored in works such as [6] and [1]. Over time, various modifications to the adjacency definition have emerged, giving rise to alternative graph representations like the Power Graph [5], Order Element Graph [8], and Coprime Graph [7].

Similar to other graph-based group representations, the coprime graph assigns group elements as vertices, with edges formed between pairs of elements whose orders are coprime. This concept was originally introduced in [7]. Further investigations into coprime graphs for dihedral groups are discussed in [12], while their application to generalized quaternion groups is elaborated in [9].

Graph theory is widely recognized for its broad applications across various disciplines, including chemistry. In this context, the concept of a connectivity index serves as a quantitative descriptor of molecular structures, derived from the associated chemical graph. Several well-known connectivity indices have gained attention, such as the Wiener index, hyper-Wiener index, Harary index, and the first and second Zagreb indices, as well as the Szeged index. These metrics have proven useful in the analysis of molecular frameworks [13] and the chemical behavior of substances like paraffin [14]. A prior study [18] explored the connectivity indices of coprime graphs over generalized quaternion groups for specific prime values of n. The present study extends that work by considering the generalized quaternion group of order $2p$ for arbitrary odd primes p, thereby contributing to a more general understanding.

2. RESEACH METHODOLOGY

This study explores the structural patterns that emerge in the coprime graphs associated with generalized quaternion groups of order $n = 2p$, where p is a prime number. According to previous findings [11], the resulting coprime graph for such a group forms a tripartite structure. However, as this graph is not a complete tripartite graph, determining the degree of each vertex becomes nontrivial. Careful analysis of these structural patterns allows for the derivation of appropriate formulas to compute the targeted connectivity indices.

Graph-theoretical terminology adopted in this research follows the conventions in [10]. Additionally, several key properties of generalized quaternion groups drawn from [9] are essential for the analysis. Prior to proceeding, the definition of the generalized quaternion group is recalled as follows:

Definition 2.1[16] The generalized quaternion group, denoted by Q_{4n}, is defined by the following group presentation:

$$Q_{4n} = \{a, b | a^{2n} = b^4 = e, b^{-1}ab = a^{-1}\}.$$

Several properties of the generalized quaternion group that are relevant to this study are outlined below.

Theorem 2.2 [9] Let Q_{4n} be a generalized quaternion group. Then the following hold:

1. $|Q_{4n}| = 4n$
2. The group Q_{4n} is Abelian if and only if $n = 1$
3. Each element of Q_{4n} can be expressed as $a^i b^j$ with $1 \leq i < n$ and $b = 0,1$.

Theorem 2.3 [9] If Q_{4n} is a generalized quaternion group, then the order of each element in the group can be determined using the following rule:

$$o(a^i b^j) = \begin{cases} \dfrac{2n}{gcd(i, 4n)}, & j = 0 \\ 4 \quad , & j = 1 \end{cases}$$

This research discusses the connectivity index on coprime graphs over generalized quaternion groups. The connectivity indices to be found include the First Zagreb Index, Second Zagreb Index, Weiner Index, Hyper-Weiner Index, Harary Index and Szeged Index. The definitions of the six indices for connected graph are as follows

Definition 2.4 [2] Let Ω be a connected graph. The **First Zagreb index**, denoted by $M_1(\Omega)$, is defined as:

$$M_1(\Omega) = \sum_{v \in V\,(\Omega)} \deg\,(v)^2$$

where $d(v)$ represents the degree of vertex v, i.e., the number of edges incident to v.

Definition 2.5 [2] Given a connected graph Ω, the **Second Zagreb index**, written as $M_2(\Omega)$, is defined by:

$$M_2(\Omega) = \sum_{uv\in E(\Omega)} deg(u)\, deg(v)$$

where $d(u)$ and $d(v)$ denote the degrees of the vertices connected by the edge uv.

Definition 2.6 [4] For a connected graph Ω, the **Wiener index**, denoted $W(\Omega)$, is given by:

$$W(\Omega) = \sum_{u,v\in V(\Omega)} d(u,v)$$

where $d(u,v)$ is the distance between vertices u and v, defined as the length of the shortest path connecting them.

Definition 2.7 [17] Let Ω be a connected graph. The **Hyper-Wiener index**, denoted by $WW(\Omega)$, is defined as:

$$WW(\Omega) = \frac{1}{2}\left(W(\Omega) + \sum_{u,v\in V(\Omega)} d(u,v)^2\right)$$

where $d(u,v)$ is the distance between vertices u and v, defined as the length of the shortest path connecting them.

Definition 2.8 [15] For a connected graph Ω, the **Harary index**, symbolized as $H(\Omega)$, is defined as:

$$H(\Omega) = \sum_{u,v\in V(\Omega)} \frac{1}{d(u,v)}$$

where $d(u,v)$ is the distance between vertices u and v.
Definition 2.9 [2] Given a simple connected graph Ω and an edge $e = uv \in E(G)$, the **Szeged index**, denoted by $Sz(G)$, is defined as:

$$Sz(\Omega) = \sum_{e\in E(\Omega)} |N_1(e|\Omega)||N_2(e|\Omega)|,$$

where $N_1(e|\Omega) = \{w \in V(\Omega) | d(w,u) < d(w,v)\}$ is the number of vertices closer to u than to v, and $N_2(e|\Omega) = \{w \in V(\Omega) | d(w,v) < d(w,u)\}$ is the number of vertices closer to v than to u.

3. MAIN RESULTS

This section is divided into two parts. The first part focuses on the structural patterns that arise in coprime graphs constructed from generalized quaternion groups. The second part is devoted to calculating various connectivity indices based on these structures.

3.1. Coprime Graph over Generalized Quaternion Group

The first part of the result is related to the coprime graph formed on the generalized quaternion group. Before proceeding further, we provide the formal definition of the coprime graph along with a brief review of several previous results concerning coprime graphs for certain group orders. This foundation is essential before focusing on the case where the group has order $n = 2p$, which will lead to the main result of this study.

Definition 3.1. [7] Let G be a group. The **coprime graph** associated with G is defined as a graph in which each element of the group corresponds to a vertex, and two vertices are connected by an edge if the orders of the corresponding elements are coprime.

Theorem 3.2.[18] Let Q_{4n} be a generalized quaternion group, and $\Omega_{Q_{4n}}$ denote its coprime graph. If n is an odd prime number, then $\Omega_{Q_{4n}}$ forms a tripartite graph.

Theorem 3.3. [18] If the generalized quaternion group Q_{4n} is such that $n = 2^k$, then the associated coprime graph $\Omega_{Q_{4n}}$ is a star graph.

Theorem 3.4. [11] For a generalized quaternion group Q_{4n} where $n = p_1^{k_1} p_2^{k_2} \dots p_l^{k_l}$, with $p_1 = 2$ and p_i are prime numbers, then then $\Omega_{Q_{4n}}$ is a $l + 1$ partite graph.

The existing literature has not yet explored certain structural properties of coprime graphs over generalized quaternion groups of specific orders. The main contribution of this work, which has not appeared in previous studies, begins with the following analysis. This research will focus on the generalized quaternion group with order $n = 2p$. Based on the previous theorem, the group with this order will form a tripartite graph.

Corollary 3.5. Let Q_{4n} be a generalized quaternion group, and $\Omega_{Q_{4n}}$ its coprime graph. If $n = 2p$, with p is an odd prime, then $\Omega_{Q_{4n}}$ is a tripartite graph.

Example 3.6. Consider the generalized quaternion group Q_{40}, defined as follows:

$$Q_{40} = \{e, a, a^2, \dots, a^{19}, b, ab, a^2b, \dots, a^{19}b\}.$$

By considering the order of each element in the group, a tripartite Coprime graph can be obtained as follows

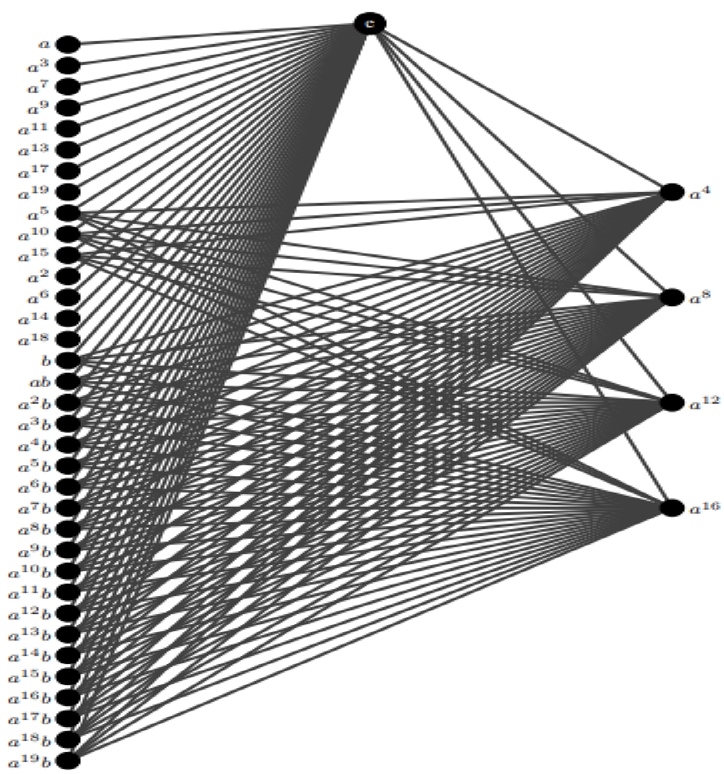

Figure1. Coprime graph over group Q_{40}

If we look at the coprime graph formed from the group Q_{40} forms an incomplete tripartite graph. This pattern will also appear for the generalized quaternion group with $n = 2p$, as p is an odd prime, with details of vertex and edge partitions as follows

$$V(\Omega_{Q_{4n}}) = P_0 \cup P_1 \cup P_2 \cup P$$

Where:

$P_0 = \{e\}$

$P_1 = [a^{4s}] = \{a^{4s} | s = 1,2,..\frac{n}{2} - 1\}$

$P_2 = \left[a^{\frac{n}{2}t}\right] \cup [a^{4s+2}] \cup [a^i b]$, With $\left[a^{\frac{n}{2}t}\right] = \{a^{\frac{n}{2}t} | t = 1,2,3, \dots, \frac{2n}{p} - 1\}$, $[a^{4s+2}] = \{a^{4s+2} | s = 1,2, \dots, \frac{n}{2} - 1\}$, $[a^i b] = \{a^i b | i = 0,1, \dots, 2n - 1\}$

$P = V(\Omega_{Q_{4n}}) - (P_0 \cup P_1 \cup P_2)$

Tripartite graph formed by selecting vertex partitions P_0, P_1 and $P_2 \cup P$. Then the edges of the coprime graph can be partitioned into

$$E(\Omega_{Q_{4n}}) = \{[a] \cup [b] \cup [c] \cup [d] \cup [f] \cup [g] \cup [h]\}$$

With

$[a] = \left\{ea^{\frac{n}{2}t} \middle| t = 1,2,3 \dots\right\}$

$[b] = \{ea^{4s+2} | s = 1,2, \dots, \frac{n}{2} - 1\}$

$[c] = \{ea^{4s} | s = 1,2,..\frac{n}{2} - 1\}$

$[d] = \left\{ea^{4s} \middle| s = 1,2, \dots, \frac{n}{2} - 1\right\}$

$[f] = \left\{ea^i b \middle| i = 0,1,2, \dots, 2n - 1\right\}$

$[g] = \{a^{\frac{n}{2}t} a^{4s} | s = 1,2, \dots, \frac{n}{2} - 1, t = 1,2,3, \dots, \frac{2n}{p}\}$

$[h] = \{a^{4s} a^i b | s = 1,2, \dots, \frac{n}{2} - 1, i = 0,1,2, \dots, 2n - 1\}$

3.2. Connectivity Index of Coprime Graphs over Generalized Quaternion Groups for $n = 2p$

By utilizing the vertex and edge structures described above, the connectivity indices of the coprime graph constructed over the generalized quaternion group can be determined for the case $n = 2p$, where p is an odd prime. The computation of these indices begins with the evaluation of the first Zagreb index, as outlined below.

Theorem 3.7. Given a coprime graph Ω over the generalized quaternion group Q_{4n}. The first Zagreb index over $\Omega_{Q_{4n}}$, for $n = 2p$ with p an odd prime number, is

$$M_1(\Omega_{Q_{4n}}) = \frac{5}{2}n^3 + \frac{83}{4}n^2 - 292n - 18.$$

Proof. As $n = 2p$ with p an odd prime number, and then the degree of each vertex of the graph $\Omega_{Q_{4n}}$ is as follows.

1. Since the vertex e is adjacent to every vertex in $\Omega_{Q_{4n}}$ then $\deg(e) = 4n - 1$.

2. The set of vertices is $\left[a^{\frac{n}{2}t}\right]$ is only adjacent to e and the set of shaped vertices $[a^{4s}]$, thus $\deg\left(a^{\frac{n}{2}t}\right) = \frac{n}{2}$, for $t = 1,2,3 \ldots, \frac{2n}{p} - 1$.
3. Since every vertex in $[a^{4s+2}]$ is only adjacent to e, then $\deg(a^{4s+2}) = 1$, for $s = 1,2,\ldots,\frac{n}{2} - 1$.
4. Since every vertex in P is only adjacent to e, then we get $\deg(p) = 1$, for every $p \in P$.
5. Every vertex in $[a^{4s}]$ is adjacent to $\{e\}$, $[a^{\frac{n}{2}t}]$, and $[a^i b]$, while the number of vertices e, $\left[a^{\frac{n}{2}t}\right]$, and $\left[a^i b\right]$ are $1, \frac{2n}{p} - 1$, and $2n$, thus $\deg(a^{4s}) = 2n + \frac{2n}{p}$, for $s = 1,2,\ldots,\frac{n}{2} - 1$
6. Every vertex in the form $[a^i b]$ is only adjacent to e and $[a^{4s}]$, while the number of vertices e is 1 and the vertex $[a^{4s}]$ is $\frac{n}{2} - 1$, so $\deg(a^i b) = \frac{n}{2}$, for $i = 0,1,2,\ldots,2n-1$.

Then, from the above conditions, the First Zagreb Index is obtained as follows,

$$\begin{aligned}
M_1(\Omega_{Q_{4n}}) &= \sum_{v \in V(\Omega_{Q_{4n}})} (\deg(v))^2 \\
&= \deg(e)^2 + \deg\left(a^{\frac{n}{2}t}\right)^2 + \sum_{v \in [a^{4s+2}]} \deg(v)^2 + \sum_{v \in P} \deg(v)^2 \\
&\quad + \sum_{v \in [a^{4s}]} \deg(v)^2 + \sum_{v \in [a^i b]} \deg(v)^2 \\
&= (4n-1)^2 + \left(\frac{2n}{p} - 1\right)\left(\frac{n}{2}\right)^2 + \left(\frac{n}{2} - 1\right) + (n-2) + \left(\frac{n}{2} - 1\right)(2n+4)^2 \\
&\quad + (2n)\left(\frac{n}{2}\right)^2.
\end{aligned}$$

Because $p = \frac{n}{2}$, then

$$\begin{aligned}
M_1(\Omega_{Q_{4n}}) &= (16n^2 - 8n + 1) + \left(\frac{3}{4}n^2\right) + \left(\frac{n}{2} - 1\right) + (n-2) + (2n^3 + 4n^2 - 8n - 16) \\
&\quad + \left(\frac{n^3}{2}\right) \\
&= \frac{5}{2}n^3 + \frac{83}{4}n^2 - \frac{29}{n} - 18.
\end{aligned}$$

Thus, the First Zagreb Index of $\Omega_{Q_{4n}}$ with $n = 2p$ where p is an odd prime number is $\frac{5}{2}n^3 + \frac{83}{4}n^2 - \frac{29}{2}n - 18$. ■

The next index investigated is the second Zagreb Index whose calculation involves multiplying the degrees of the connected vertices.

Theorem 3.8. Given a coprime graph Ω over generalized quaternion group Q_{4n}. Second Zagreb index over $\Omega_{Q_{4n}}$ is

$$M_2(\Omega_{Q_{4n}}) = n^4 + \frac{19}{2}n^3 + 6n^2 - 37n + 7,$$

if $n = 2p$ with p is an odd prime number.

Proof. Given $n = 2p$ with p is an odd prime number. Based on the proof in Theorem 3.7, we get

$$M_2(\Omega_{Q_{4n}}) = \sum_{u,v\in E(\Omega_{Q_{4n}})} \deg(u)\deg(v)$$
$$= \deg(e)\deg\left(a^{\frac{n}{2}t}\right) + \sum_{uv\in[b]} \deg(u)\deg(v) + \sum_{uv\in[c]} \deg(u)\deg(v)$$
$$+ \sum_{uv\in[d]} \deg(u)\deg(v) + \sum_{uv\in[f]} \deg(u)\deg(v) + \sum_{uv\in[g]} \deg(u)\deg(v)$$
$$+ \sum_{uv\in[h]} \deg(u)\deg(v)$$
$$= (4n-1)\left(\frac{2n}{p}-1\right)\left(\frac{n}{2}\right) + (4n-1)\left(\frac{n}{2}-1\right) + (4n-1)\left(n-\frac{2n}{p}+2\right)$$
$$+ (4n-1)\left(\frac{n}{2}-1\right)\left(2n+\frac{2n}{p}\right) + (4n-1)(2n)\left(\frac{n}{2}\right)$$
$$+ \left(\frac{2n}{p}-1\right)\left(\frac{n}{2}-1\right)\left(\frac{n}{2}\right)\left(2n+\frac{2n}{p}\right) + \left(\frac{n}{2}-1\right)(2n)\left(2n+\frac{2n}{p}\right)\left(\frac{n}{2}\right)$$

because $p = \frac{n}{2}$, then

$$M_2(\Omega_{Q_{4n}}) = \left(6n^2 - \frac{3}{2}n\right) + \left(2n^2 - \frac{9}{2}n + 1\right) + (4n^2 - 9n + 2) + (4n^3 - n^2 - 16n + 4)$$
$$+ (4n^3 - n^2) + \left(\frac{3}{2}n^3 - 6n\right) - (n^4 - 4n^2)$$
$$= n^4 + \frac{19}{2}n^3 + 6n^2 - 37n + 7.$$

So the Second Zagreb Index over $\Omega_{Q_{4n}}$ for $n = 2p$is $n^4 + \frac{19}{2}n^3 + 6n^2 - 37n + 7$. ■

The next indices analyzed are Weiner and Hyper Weiner. The calculation of these two indexes involves the distance between each vertex in the graph that is adjacent to each other.

Theorem 3.9. Given a coprime graph Ω over generalized quaternion group Q_{4n}. The Wiener index of $\Omega_{Q_{4n}}$ is

$$W(\Omega_{Q_{4n}}) = 16n^2 - \frac{47}{2}n + 8,$$

with $n = 2p$, where p is an odd prime number

Proof. Given $n = 2p$ with p is another prime number, we get

$$W(\Omega_{Q_{4n}}) = \sum_{u,v\in V(\Omega_{Q_{4n}})-\{e\}} d(u,v) + \sum_{u\in a^{\frac{n}{2}t}, v\in P_1} d(u,v) + \sum_{u\in[a^{4s+2}], v\in\{P_2\cup P\}} d(u,v) + \sum_{u\in P, v\in\{P_1\cup P_2\}} d(u,v)$$
$$+ \sum_{u.\in P_1, v\in[a^i b]} d(u,v) + \sum_{u,v\in\left[a^{\frac{n}{2}t}\right]} d(u,v) + \sum_{u,v\in[a^{4s+2}]} d(u,v) + \sum_{u,v\in P} d(u,v)$$
$$+ \sum_{u,v\in[a^{4s}]} d(u,v) + \sum_{u,v\in[a^i b]} d(u,v)$$

$$= (4n-1) + 3\left(\frac{n}{2}-1\right) + 2\left(\frac{n}{2}-1\right)\left(\frac{7n}{2}\right) + 2(n-2)(1+3n) + \left(\frac{n}{2}-1\right)(2n) + 2\binom{3}{2} + 2\binom{\frac{n}{2}-1}{2}$$
$$+ 2\binom{n-2}{2} + 2\binom{\frac{n}{2}-1}{2} + 2\binom{2n}{2}$$
$$= (4n-1) + (3p-3) + \left(\frac{7n^2}{2} - 7n\right) + (6n^2 + 6n - 32p - 4) + (n^2 - 2n)$$
$$+ \left(\frac{4n^2}{p^2} - 10\right) + \left(\frac{n^2}{4} - \frac{3n}{2} + 2\right) + (n^2 + 3n - 16p + 6) + \left(\frac{n^2}{4} - \frac{3n}{2} + 2\right)$$
$$+ (4n^2 - 2n)$$

Because $p = \frac{n}{2}$, then

$$W(\Omega_{Q_{4n}}) = 16n^2 - \frac{47}{2}n + 8.$$

So the Wiener Index of $\Omega_{Q_{4n}}$ with $n = 2p$ is $16n^2 - \frac{47}{2}n + 8$. ■

Theorem 3.10. Given a coprime graph Ω over the generalized quaternion group Q_{4n}. *The hyper-Wiener* index over $\Omega_{Q_{4n}}$ with $n = 2p$ and p is an odd prime number is

$$WW(\Omega_{Q_{4n}}) = \frac{47}{2}n^2 - 37n + 14.$$

Proof. Given $n = 2p$, with p is another prime number. Let us first determine the sum of squared distances between two vertices in $\Omega_{Q_{4n}}$, as follows

$$\sum_{u,v\in V(\Omega)} d(u,v)^2 = \sum_{u,v\in V(\Omega_{Q_{4n}})-\{e\}} d(u,v)^2 + \sum_{u\in a^{\frac{n}{2}t},v\in P_1} d(u,v)^2 + \sum_{u\in[a^{4s+2}],v\in\{P_2\cup P\}} d(u,v)^2$$
$$+ \sum_{u\in P,v\in\{P_1\cup P_2\}} d(u,v)^2 + \sum_{u.\in P_1,v\in[a^ib]} d(u,v)^2 + \sum_{u,v\in\left[a^{\frac{n}{2}t}\right]} d(u,v)^2 + \sum_{u,v\in[a^{4s+2}]} d(u,v)^2$$
$$+ \sum_{u,v\in P} d(u,v)^2 + \sum_{u,v\in[a^{4s}]} d(u,v)^2 + \sum_{u,v\in[a^ib]} d(u,v)^2$$
$$= (4n-1) + \left(\frac{2n}{p}-1\right)\left(\frac{n}{2}-1\right) + \left(\frac{n}{2}-1\right)\left(\frac{7n}{2}\right)2^2 + (n-2)(1+3n)2^2 + \left(\frac{n}{2}-1\right)2n$$
$$+ \binom{3}{2}2^2 + \binom{\frac{n}{2}-1}{2}2^2 + \binom{n-2}{2}2^2 + \binom{\frac{n}{2}-1}{2}2^2 + \binom{2n}{2}2^2$$
$$= (4n-1) + (3p-3) + (7n^2 - 14n) + (12n^2 + 12n - 64p - 8) + (n^2 - 2n)$$
$$+ \left(\frac{8n^2}{p^2} - 20\right) + \left(\frac{n^2}{2} - 3n + 4\right) + (2n^2 + 6n - 32p + 12) + \left(\frac{n^2}{2} - 3n + 4\right)$$
$$+ (8n^2 - 4n)$$

Because $p = \frac{n}{2}$, then obtained

$$\sum_{u,v\in V(\Omega)} d(u,v)^2 = 31n^2 - \frac{101}{2}n + 20.$$

Based on Theorem 3.9, we obtain

$$\begin{aligned}WW(\Omega_{Q_{4n}}) &= \frac{1}{2}\left(W(\Omega_{Q_{4n}}) + \sum_{u,v\in V(\Omega_{Q_{4n}})} \left(d(u,v)\right)^2\right)\\ &= \frac{1}{2}\left(\left(16n^2 - \frac{47}{2}n + 8\right) + \left(31n^2 - \frac{101}{2}n + 20\right)\right)\\ &= \frac{47}{2}n^2 - 37n + 14.\end{aligned}$$

Thus, the *hyper-Wiener* index over $\Omega_{Q_{4n}}$ with $n = 2p$, where p is an odd prime number is $\frac{47}{2}n^2 - 37n + 14$. ■

Theorem 3.11. Given a coprime graph Ω over the generalized quaternion group Q_{4n}. The Harary index over $\Omega_{Q_{4n}}$ is

$$H(\Omega_{Q_{4n}}) = \frac{19}{4}n^2 - \frac{13}{4}n - 1,$$

with $n = 2p$ and p odd primes.

Proof. Harary Index is the sum of the inverse of the distance between two vertices in the $\Omega_{Q_{4n}}$, so using the result in Theorem 3.10, the Harary Index is obtained as follows

$$\begin{aligned}H(\Omega_{Q_{4n}}) = &\sum_{u=e,v\in V(\Omega_{Q_{4n}})-\{e\}} \frac{1}{d(u,v)} + \sum_{u\in\left[a^{\frac{n}{2}t}\right],v\in P_1} \frac{1}{d(,v)}\\ &+ \sum_{u\in[a^{4s+2}],v\in\left\{\left[a^{\frac{n}{2}t}\right]\cup P\cup[a^{4s}]\cup[a^ib]\right\}} \frac{1}{d(u,v)} + \sum_{u\in P,v\in P_1\cup p_2} \frac{1}{d(u,v)}\\ &+ \sum_{u\in P_1,v\in[a^ib]} \frac{1}{d(u,v)} + \sum_{u,v\in\left[a^{\frac{n}{2}t}\right]} \frac{1}{d(u,v)} + \sum_{u,v\in[a^{4s+2}]} \frac{1}{d(u,v)} + \sum_{u,v\in P} \frac{1}{d(u,v)}\\ &+ \sum_{u,v\in[a^{4s}]} \frac{1}{d(u,v)} + \sum_{u,v\in[a^ib]} \frac{1}{d(u,v)}\\ = &(4n-1) + \left(\frac{2n}{p} - 1\right)\left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} - 1\right)\left(\frac{7n}{2}\right)\frac{1}{2}\\ &+ \left(n - \frac{2n}{p} + 2\right)\left(\frac{2n}{p} - 3 + 3n\right)\frac{1}{2} + \left(\frac{n}{2} - 1\right)2n + \binom{\frac{2n}{p} - 1}{2}\frac{1}{2}\\ &+ \binom{n - \frac{2n}{p} + 2}{2}\frac{1}{2} + \binom{\frac{n}{2} - 1}{2}\frac{1}{2} + \binom{2n}{2}\frac{1}{2}.\end{aligned}$$

Because $p = \frac{n}{2}$, then

$$\begin{aligned}H(\Omega_{Q_{4n}}) = &(4n-1) + \left(\frac{3}{2}n - 3\right) + \left(\frac{7}{8}n^2 - \frac{7}{4}n\right) + \left(\frac{3}{2}n^2 + \frac{5}{2}n - 1\right) + (n^2 - 2n) + \left(4 - \frac{5}{2}\right)\\ &+ \left(\frac{n^2}{16} - \frac{3}{8}n + \frac{1}{2}\right) + \left(\frac{n^2}{4} - \frac{5}{4}n + \frac{3}{2}\right) + \left(\frac{n^2}{16} - \frac{3}{8}n + \frac{1}{2}\right) + \left(n^2 - \frac{n}{2}\right)\end{aligned}$$

$$= \frac{19}{4}n^2 - \frac{13}{4}n - 1.$$

Thus, the Harary Index of $\Omega_{Q_{4n}}$ with $n = 2p$, where p is a prime odd number, is $\frac{19}{4}n^2 + \frac{13}{4}n - 1$. ■

Theorem 3.22. Given a coprime graph $\Omega_{Q_{4n}}$ over the generalized quaternion group. The Szeged index of $\Omega_{Q_{4n}}$ is

$$Sz(\Omega_{Q_{4n}}) = n^4 - n^3 + \frac{5}{4}n^2 + 8n - 5,$$

if $n = 2p$ with p is another prime number.

Proof. Given $n = 2p$, where p is another prime number. We can determine the distance of *adjacent* vertices in $\Omega_{Q_{4n}}$, as follows

1. Edge $a = ea^{\frac{n}{2}t}$
   $N_1(a|\Omega_{Q_{4n}}) = \left\{u \in V(\Omega_{Q_{4n}}) \middle| d(u,e) < d\left(u, \left[a^{\frac{n}{2}t}\right]\right)\right\} = [a^{4s+2}] \cup P \cup [a^i b],$
   $N_2(a|\Omega_{Q_{4n}}) = \left\{u \in V(\Omega_{Q_{4n}}) \middle| d\left(u, \left[a^{\frac{n}{2}t}\right]\right) < d(u,e)\right\} = \left\{a^{\frac{n}{2}t}\right\},$
   so $N_1(a|\Omega_{Q_{4n}}) = \left(\frac{n}{2} - 1\right) + \left(n - \frac{2n}{p} + 2\right) + (2n) = \frac{7}{2}n - \frac{2n}{p} + 1$ and $N_2(a|\Omega_{Q_{4n}}) = 1$.
2. Edge $b = ea^{4s+2}$
   $N_1(b|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}}) | d(u,e) < d(u,[a^{4s+2}])\} = \left[a^{\frac{n}{2}t}\right] \cup P \cup [a^{4s}] \cup [a^i b]$
   $N_2(b|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}}) | d(u,[a^{4s+2}]) < d(u,e)\} = \{a^{4s+2}\},$
   so $N_1(b|\Omega_{Q_{4n}}) = \left(\frac{2n}{p} - 1\right) + \left(n - \frac{2n}{p} + 2\right) + \left(\frac{n}{2} - 1\right) + (2n) = \frac{7}{2}n$ and $N_2(b|\Omega_{Q_{4n}}) = 1$.
3. Edge $c = eP$
   $N_1(c|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}}) | d(u,e) < d(u,P)\} = \left[a^{\frac{n}{2}t}\right] \cup [a^{4s+2}] \cup [a^{4s}] \cup [a^i b],$
   $N_2(c|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}}) | d(u,P) < d(u,e)\} = P,$
   so $N_1(c|\Omega_{Q_{4n}}) = \left(\frac{2n}{p} - 1\right) + \left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} - 1\right) + 2n = 3n + \frac{2n}{p} - 3$ and $N_2(c|\Omega_{Q_{4n}}) = 1$.
4. Edge $d = ea^{\frac{2n}{p}s}$
   $N_1(d|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}}) | d(u,e) < d(u,[a^{4s}])\} = [a^{4s+2}] \cup [P],$
   $N_2(d|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}}) | d(u,[a^{4s}]) < d(u,e)\} = \{a^{4s}\},$
   so $N_1(d|\Omega_{Q_{4n}}) = \left(\frac{n}{2} - 1\right) + (n - 4 + 2) = \frac{3}{2}n - 3$ and $N_2(d|\Omega_{Q_{4n}}) = 1$.
5. Edge $f = ea^i b$
   $N_1(f|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}}) | d(u,e) < d(u,[a^i b])\} = \left[a^{\frac{n}{2}t}\right] \cup [a^{4s+2}] \cup [P],$
   $N_2(f|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}}) | d(u,[a^i b]) < d(u,e)\} = \{a^i b\},$
   so $N_1(f|\Omega_{Q_{4n}}) = \left(\frac{2n}{p} - 1\right) + \left(\frac{n}{2} - 1\right) + \left(n - \frac{2n}{p} + 2\right) = \frac{3}{2}n$ and $N_2(f|\Omega_{Q_{4n}}) = 1$.
6. Edge $g = a^{\frac{n}{2}t}a^{4s}$
   $N_1(g|\Omega_{Q_{4n}}) = \left\{u \in V(\Omega_{Q_{4n}}) \middle| d\left(u, \left[a^{\frac{n}{2}t}\right]\right) < d(u,[a^{4s}])\right\} = \{a^{4j} | j \le s\} \cup \left[a^{\frac{n}{2}t}\right],$
   $N_2(g|\Omega_{Q_{4n}}) = \left\{u \in V(\Omega_{Q_{4n}}) \middle| d(u,[a^{4s}]) < d\left(u, \left[a^{\frac{n}{2}t}\right]\right)\right\} = \{a^{4s}\} \cup [a^i b],$

so $N_1\left(g\middle|\Omega_{Q_{4n}}\right) = \left(\frac{n}{2} - 1 - 1\right) + 1 = \frac{n}{2} - 1$ and $N_2\left(g\middle|\Omega_{Q_{4n}}\right) = 1 + 2n$.

7. Edge $h = a^{4s}a^i b$

$N_1\left(h\middle|\Omega_{Q_{4n}}\right) = \left\{u \in V\left(\Omega_{Q_{4n}}\right)\middle|d\left(u,[a^{4s}]\right) < d\left(u,\left[a^i b\right]\right)\right\} = \left\{a^j b\middle|j \leq i\right\} \cup \left[a^{\frac{n}{2}t}\right] \cup \{a^{4s}\}$,

$N_2\left(h\middle|\Omega_{Q_{4n}}\right) = \left\{u \in V\left(\Omega_{Q_{4n}}\right)\middle|d\left(u,\left[a^i b\right]\right) < d\left(u,[a^{4s}]\right)\right\} = \left\{a^{4j}\middle|j \leq s\right\} \cup \left\{a^i b\right\}$,

so $N_1\left(h\middle|\Omega_{Q_{4n}}\right) = 3 + 1 + (2n - 1) = 2n + 3$ and $N_2\left(h\middle|\Omega_{Q_{4n}}\right) = \left(\frac{n}{2} - 1 - 1\right) + 1 = \frac{n}{2} - 1$.

Based on the explanation above, the Szeged Index of the graph is as follows

$$\begin{aligned}
Sz\left(\Omega_{Q_{4n}}\right) &= \sum_{u \in E\left(\Omega_{Q_{4n}}\right)} \left|N_1\left(u\middle|\Omega_{Q_{4n}}\right)\right| \left|N_2\left(u\middle|\Omega_{Q_{4n}}\right)\right| \\
&= \left|N_1\left(a\middle|\Omega_{Q_{4n}}\right)\right|\left|N_2\left(a\middle|\Omega_{Q_{4n}}\right)\right| + \sum_{b \in [b]} \left|N_1\left(b\middle|\Omega_{Q_{4n}}\right)\right|\left|N_2\left(b\middle|\Omega_{Q_{4n}}\right)\right| \\
&\quad + \sum_{c \in [c]} \left|N_1\left(c\middle|\Omega_{Q_{4n}}\right)\right|\left|N_2\left(c\middle|\Omega_{Q_{4n}}\right)\right| + \sum_{d \in [d]} \left|N_1\left(d\middle|\Omega_{Q_{4n}}\right)\right|\left|N_2\left(d\middle|\Omega_{Q_{4n}}\right)\right| \\
&\quad + \sum_{f \in [f]} \left|N_1\left(f\middle|\Omega_{Q_{4n}}\right)\right|\left|N_2\left(f\middle|\Omega_{Q_{4n}}\right)\right| + \sum_{g \in [g]} \left|N_1\left(g\middle|\Omega_{Q_{4n}}\right)\right|\left|N_2\left(g\middle|\Omega_{Q_{4n}}\right)\right| \\
&\quad + \sum_{h \in [h]} \left|N_1\left(h\middle|\Omega_{Q_{4n}}\right)\right|\left|N_2\left(h\middle|\Omega_{Q_{4n}}\right)\right| \\
&= \left(\frac{2n}{p} - 1\right)\left(\frac{7}{2}n - \frac{2n}{p} + 1\right) + \left(\frac{n}{2} - 1\right)\left(\frac{7}{2}n\right) + \left(n - \frac{2n}{p} + 2\right)\left(3n + \frac{2n}{p} - 3\right) \\
&\quad + \left(\frac{n}{2} - 1\right)\left(\frac{3}{2}n - \frac{2n}{p} + 1\right) + 2n\left(\frac{3}{2}n\right) + \left(\frac{2n}{p} - 1\right)\left(\frac{n}{2} - 1\right)(1 + 2n) \\
&\quad + \left(\frac{n}{2} - 1\right)2n + \left(2n + \frac{2n}{p} - 1\right)\left(\frac{n}{2} - 1\right)
\end{aligned}$$

because $p = \frac{n}{2}$, then

$$Sz\left(\Omega_{Q_{4n}}\right) = n^4 - n^3 + \frac{5}{4}n^2 + 8n - 5.$$

So the Szeged Index of $\Omega_{Q_{4n}}$ with $n = 2p$ where p is an odd prime number is $n^4 - n^3 + \frac{5}{4}n^2 + 8n - 5$. ■

4. CONCLUSION

By analyzing the structural patterns found in coprime graphs of generalized quaternion groups with order $n = 2p$, where p is an odd prime, six connectivity indices can be evaluated. These include the Wiener Index, Harary Index, Hyper-Wiener Index, First Zagreb Index, Second Zagreb Index, and Szeged Index.

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