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Parameter Estimation of Zero Inflated Bivariate Ordered Probit Model with Berndt, Hall, Hall, and Hausman Iteration Approach

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Abstract

Probit regression is a statistical analysis method used to analyze the relationship between response variables and predictor variables where the response variable is categorical with a normal distribution link function. Based on the measurement scale, probit regression is divided into two, namely binary probit regression and ordinal probit regression. Based on the number of response variables, ordinal probit regression is divided into two, namely univariate ordinal probit regression and multivariate ordinal probit regression. Multivariate ordinal probit regression that has two response variables is called bivariate ordinal probit regression. In univariate ordinal regression, if there are many unequal proportions in certain categories, conventional univariate probit ordinal regression cannot provide good estimation results. Therefore, univariate ordinal probit regression must be developed into Zero Inflated Ordered Probit (ZIOP) Regression. Similar to univariate ordinal probit regression, bivariate ordinal probit regression produces poor estimates if the response variable is zero inflated, so it is developed into Zero Inflated Bivariate Ordered Probit Regression (ZIBOPR). This study aims to estimate the parameters of the ZIBOPR model, using the Maximum Likelihood Estimator (MLE) method with Brendt, Hall, Hall, and Hausman (BHHH) numerical iteration. This study produces a parameter estimator of the ZIBOPR model, which is a combination of binary probit regression and bivariate ordinal probit regression with the BHHH numerical iteration approach.

Keywords: ZIBOPR, MLE, BHHH.

1. INTRODUCTION

Probit regression is a statistical analysis method used to analyze the relationship between response variables and predictor variables where the response variable is categorical with a normal distribution link function. Based on the data measurement scale, probit regression is divided into two, namely binary probit regression and ordinal probit regression. In probit regression which has two categories in the response variable is called binary probit regression. However, data categories are not always only two categories. Therefore, the development of binary probit regression is ordinal probit regression which has categories in the response variable that are more than two categories and have levels [12].

Ordinal probit regression is a regression method used to see the relationship between predictor response variables where the response variable is more than two categories and where the categories have levels [10]. Ordinal probit regression that has more than one response and is correlated is called multivariate probit regression [5]. In multivariate ordinal probit regression, there are only two response variables in one ordinal probit regression model called bivariate ordinal probit regression where this method is useful to see the relationship between two categorical response variables and several predictor variables that are thought to have an effect together [2].

Ordinal probit regression is effective when modeling data with ordinal response variables with frequencies in each category that tend to be balanced . However, many events in the field show an excess frequency in one category or what is called zero-inflated. Conventional ordinal probit regression that models each category together is unable to accommodate these conditions. In this case, the ordinal probit regression is developed into Zero Inflated Ordered Probit (ZIOP) regression, where ZIOP models the inflated category and other ordinal categories separately [1].

The ZIOP regression model describes an extension of the ordinal probit regression model to consider where zeros can arise from two different variables in individual treatments. The ZIOP regression model has an inflated variable which is a predictor variable that affects the chance of zero inflation where the proportion of zeros is higher than expected. Such variables are usually expressed in the form of binary probit regression. Predictor variables in ordinal probit regression are used to predict the ordinal category of the response variable [11]. The combination of binary probit regression and ordinal probit regression allows the ZIOP regression to account for zero inflation so that the predictors and the level of the response variable are linked together [6].

Bivariate ordinal probit regression produces poor estimates if the response variable is zero-inflated. Therefore, in these cases, the bivariate ordinal probit regression method is not good to use because it will produce biased parameter estimates, so it was developed into Zero Inlfated Bivariate Ordered Probit Regression (ZIBOPR) which is ZIBOPR development of the ZIOP model, which has two response variables that have levels and are zero-inflated and the two response variables are correlated [3].

ZIBOPR is a statistical method used to see the relationship between two ordinal categorical response variables where the two response variables are zero-inflated and correlated. ZIBOPR is a combination of binary probit regression and bivariate ordinal probit regression. Binary probit regression models the probability of observations with zero category or over-frequency category in handling zero-inflated and bivariate ordinal probit regression models two response variables that are correlated in data that are not zero-inflated [7].

The study conducted by Gurmu and Dagne employed the Zero-Inflated Bivariate Ordered Probit Regression (ZIBOPR) model to address excess zeros in bivariate cases, using household tobacco survey data from Bangladesh. The parameters of the ZIBOPR model were estimated using a Bayesian estimator with the Markov Chain Monte Carlo (MCMC) approach to obtain the posterior distribution of the parameters. The response variables in their study were smoking status and chewing tobacco status. The proportion in the low category was 76% for smoking status and 87% for chewing tobacco status. The results indicated that the ZIBOPR model performed better than the conventional bivariate ordered probit regression model. However, in the present study, a different estimation method is employed, namely Maximum Likelihood Estimation (MLE) followed by the BHHH numerical iteration. The likelihood function developed in Gurmu and Dagne (2012) serves as a reference for this research [7].

The BHHH iteration method is a numerical iteration method that is a development of the Newton-Raphson iteration method. BHHH maximizes the likelihood function in the parameter estimation process using the MLE method. The BHHH iteration method is simpler than Newton-Raphson because the calculation of the Hessian matrix uses a more complex second derivative, while the BHHH numerical iteration uses the Fisher Information Matrix which only uses the first derivative [16]. In addition, BHHH Iteration has an advantages in iteration speed, compared to other numerical iterations, BHHH produce faster computation time [9].

This study advances the bivariate ordinal probit regression model by formulating a zero-inflated bivariate ordered probit approach, wherein the response variables are characterized by an excess of outcome categories, exhibit inherent ordering, and display interdependence through mutual correlation**.** This study will examine how the parameter estimation process in the Zero Inflated Bivariate Ordered Probit Regression (ZIBOPR) model using the Maximum Likelihood Estimator (MLE) estimation method. However, the gradient of the log-likelihood function produces an equation that is not closeform, so the parameter estimation is continued with BHHH numerical iteration. Before parameter estimation, it is essential to conduct multicollinearity diagnostics on all predictor variables to ensure the absence of strong linear relationships among the independent variables, thereby enhancing the reliability and validity of the estimation results.

2. ZERO INFLATED BIVARIATE ORDERED PROBIT

The Zero Inflated Bivariate Ordered Probit Regression (ZIBOPR) model is a development of the ZIOP regression model in which there are two response variables in ZIBOPR that are correlated with each other and the two response variables observed have levels or ordinal and use a bivariate normal distribution link function. Like the case of ZIOP regression, both response variables in ZIBOPR have inflation in category 0. In the case of correlated variables, ZIOP regression is not appropriate because correlated variables will lead to poor estimation results.

In ZIBOPR, suppose Y_1^* and Y_2^* are two unobservable response variables corresponding to the observed ordinal response variables, namely Y_1 and Y_2 where the ordered ordinal values are $0, 1, \dots, J$ and $0, 1, \dots, K$ [14]. Assume that the response variable $\boldsymbol{Y}^* = (Y_1^*, Y_2^*)$ follows the ZIBOPR model which is shown in the following equation [15].

$$\begin{aligned} Y_1^* &= \boldsymbol{z}_1^{\boldsymbol{T}}\boldsymbol{\alpha}_1 + \varepsilon_1 \\ Y_2^* &= \boldsymbol{z}_2^{\boldsymbol{T}}\boldsymbol{\alpha}_2 + \varepsilon_2 \end{aligned} \quad (2.1)$$

where,

$\boldsymbol{z}_1$: Vector of predictor variables for Y_1^* with $\boldsymbol{z}_1 = [1 \quad Z_{11} \quad \dots \quad Z_{1p}]^T$
$\boldsymbol{z}_2$: Vector of predictor variables for Y_2^* with $\boldsymbol{z}_2 = [1 \quad Z_{21} \quad \dots \quad Z_{2p}]^T$
$\boldsymbol{\alpha}_1$ **:** Vector of regression parameters Y_1^* with $\boldsymbol{\alpha}_1 = [\alpha_{10} \quad \alpha_{11} \quad \dots \quad \alpha_{1p}]^T$
$\boldsymbol{\alpha}_2$ **:** Vector of regression parameters Y_2^* with $\boldsymbol{\alpha}_2 = [\alpha_{20} \quad \alpha_{21} \quad \dots \quad \alpha_{2p}]^T$
ε_1 : Error for Y_1^*
ε_2 : Error for Y_2^*

ε_1 and ε_2 are interconnected errors in the response variable Y_1^* and Y_2^* which is denoted as a vector $\boldsymbol{\varepsilon}_i = (\varepsilon_1, \varepsilon_2)^T$ which is assumed to be bivariate normal distribution with variance σ_1^2 and σ_1^2 and the correlation between the errors are denoted by $\boldsymbol{\rho}$ or denoted as follows [16].

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \sim N_2 \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right]$$

The probability of an observation being zero-inflated in ZIBOPR is described in the following equation [4].

$$P\,(s = 1|x) = P(\delta^* > 0|x) = \Phi(\mathbf{x}^{\mathbf{T}}\boldsymbol{\lambda}) \quad (2.2)$$

where,

$\Phi(.)$: cumulative distribution function of the standard normal distribution and $P\ (s = 0|x) = 1 - P\ (s = 1|x)$.

Latent variables Y_1^* and Y_2^* in the ZIBOPR model in equation (2.1) cannot be determined directly, so it uses certain thresholds, for example δ and γ. The ordinal probit categorization in ZIBOPR is denoted by r_1 and r_2 [13].
$Y_1^* \leq \delta_0$ and $Y_2^* \leq \gamma_0$ respectively for $r_1 = 0$ and $r_2 = 0$
$\delta_{j-1} < Y_1^* < \delta_j$ and $\gamma_{k-1} < Y_2^* < \gamma_k$ respectively for $r_1 = 1, \dots, j-1$ and $r_2 = 1, \dots, k-1$
$Y_1^* > \delta_{j-1}$ and $Y_2^* > \gamma_{k-1}$ respectively for $r_1 = J$ and $r_2 = K$
where J and K are the highest categories in bivariate ordinal probit, respectively. If there are cases that have four categories,
$Y_1^* \leq \delta_0$ and $Y_2^* \leq \gamma_0$ respectively for $r_1 = 0$ and $r_2 = 0$
$\delta_0 < Y_1^* < \delta_1$ and $\gamma_0 < Y_2^* < \gamma_1$ respectively for $r_1 = 1$ and $r_2 = 1$
$\delta_1 < Y_1^* < \delta_2$ and $\gamma_1 < Y_2^* < \gamma_2$ respectively for $r_1 = 2$ and $r_2 = 2$
$Y_1^* > \delta_2$ and $Y_2^* > \gamma_2$ respectively for $r_1 = 3$ and $r_2 = 3$
where δ_j and γ_k are, respectively, the threshold value of each tier of the Y_1^* and Y_2^*, with constraints $\delta_0 < \delta_1 < \delta_2 < \cdots < \delta_{j-1}$ and $\gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{k-1}$.

Variables Y_1^* and Y_2^* are not directly observable so they use the previously discussed thresholds of δ and γ. The mapping between r_1 and r_2 with Y_1^* and Y_2^* is generally given by [15],

$$r_1(Y_1^*) = \begin{cases} 0, & Y_1^* \leq \delta_0 \\ j, & \delta_{j-1} < Y_1^* < \delta_j \\ J, & Y_1^* > \delta_{j-1} \end{cases} \quad \text{and} \quad r_2(Y_2^*) = \begin{cases} 0, & Y_2^* \leq \gamma_0 \\ j, & \gamma_{k-1} < Y_2^* < \gamma_k \\ J, & Y_2^* > \gamma_{k-1} \end{cases} \tag{2.3}$$

Mapping between categories if it has four categories, r_1 and r_2 with Y_1^* and Y_2^* is given by,

$$r_1(Y_1^*) = \begin{cases} 0, & Y_1^* \leq \delta_0 \\ 1, & \delta_0 < Y_1^* < \delta_1 \\ 2, & \delta_1 < Y_1^* < \delta_2 \\ 3, & Y_1^* > \delta_2 \end{cases} \quad \text{and} \quad r_2(Y_2^*) = \begin{cases} 0, & Y_2^* \leq \gamma_0 \\ 1, & \gamma_0 < Y_2^* < \gamma_1 \\ 2, & \gamma_1 < Y_2^* < \gamma_2 \\ 3, & Y_2^* > \gamma_2 \end{cases} \tag{2.4}$$

Based on equation (2.3), the bivariate ordinal probit regression model on ZIBOPR is described in equation (2.5).

$$\begin{aligned} &P(r_1 = 0, r_2 = 0) = \Phi_2(-\mathbf{z}_1^{\boldsymbol{T}}\boldsymbol{\alpha}_1, -\mathbf{z}_2^{\boldsymbol{T}}\boldsymbol{\alpha_2}, \rho) \\ &\vdots \\ &P(r_1 = j, r_2 = k) \\ &\qquad = \Phi_2\left(\delta_{j-1} - \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\alpha}_1, \gamma_{k-1} - \mathbf{z}_2^{\mathrm{T}}\boldsymbol{\alpha}_2, \rho\right) \\ &\qquad - \Phi_2\left(\delta_j - \mathbf{z}_1^{\mathbf{T}}\boldsymbol{\alpha}_1, \gamma_k - \mathbf{z}_2^{\mathrm{T}}\boldsymbol{\alpha}_2, \rho\right); j = 1, \dots, J-1, \\ &\qquad\qquad\qquad\qquad k = 1, \dots, K-1 \\ &\vdots \\ &P(r_1 = J, r_2 = K) = 1 - \Phi_2(\delta_{J-1} - \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\alpha_1}, \gamma_{K-1} - \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\alpha}_1, \rho) \end{aligned} \tag{2.5}$$

The joint probability of having four categories in the ordinal probit section is described as (2.81).

$$\begin{aligned} P(r_1 = 0, r_2 = 0) &= \Phi_2(\delta_0 - \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\alpha}_1, \gamma_0 - \mathbf{z}_2^{\mathrm{T}}\boldsymbol{\alpha}_2, \rho) \\ P(r_1 = 0, r_2 = 1) &= \Phi_2\left(\delta_0 - \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\alpha}_1, \gamma_1 - \mathbf{z}_2^{\mathrm{T}}\boldsymbol{\alpha}_2, \rho\right) - \Phi_2\left(\delta_0 - \mathbf{z}_1^{\mathbf{T}}\boldsymbol{\alpha}, \gamma_0 - \mathbf{z}_2^{\mathrm{T}}\boldsymbol{\alpha}_2, \rho\right) \\ P(r_1 = 0, r_2 = 2) &= \Phi_2\left(\delta_0 - \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\alpha}_1, \gamma_2 - \mathbf{z}_2^{\mathrm{T}}\boldsymbol{\alpha}_2, \rho\right) - \Phi_2\left(\delta_0 - \mathbf{z}_1^{\mathbf{T}}\boldsymbol{\alpha}_1, \gamma_1 - \mathbf{z}_2^{\mathrm{T}}\boldsymbol{\alpha}_2, \rho\right) \\ P(r_1 = 0, r_2 = 3) &= 1 - \Phi_2(\delta_0 - \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\alpha_1}, \gamma_2 - \mathbf{z_2}^{\mathrm{T}}\boldsymbol{\alpha_2}, \rho) \end{aligned} \tag{2.6}$$

$$\vdots$$
$$P(r_1 = 3, r_2 = 3) = 1 - \Phi_2(\delta_3 - \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\alpha}_1, \gamma_3 - \mathbf{z}_2^{\mathrm{T}}\boldsymbol{\alpha}_2, \rho)$$

The relationship between s and (r_1, r_2) in relation to zero is described in the following equation [8].

$$(y_1, y_2) = s \times (r_1, r_2) = \begin{cases} (0,0), & s = 0 \text{ or } r_1 = 0, r_2 = 0 \\ & \vdots \\ (1,1), & s = 1 \text{ or } r_1 = 1, r_2 = 1 \\ & \vdots \\ (J, K), & s = 1 \text{ or } r_1 = J, r_2 = K \end{cases} \quad (2.7)$$

So, the general form of the odds of ZIBOPR regression is described in the following equation.

$$p(y_1, y_2) = \begin{cases} P(s = 0) + P(s = 1)P(r_1 = 0, r_2 = 0); \text{For } (y_1, y_2) = (0,0) \\ P(s = 1)P(r_1 = j, r_2 = k); \text{For } (y_1, y_2) \neq (0,0) \end{cases} \quad (2.8)$$

where $j = 0,1, \dots, J$ and $k = 0, 1, \dots, K$.

Based on equation (2.8), the ZIBOPR model is described in the following equation [7].

$$P(y_1 = 0, y_2 = 0|s = 1) = \Phi(\mathbf{x}^{\mathbf{T}}\boldsymbol{\lambda})[\Phi_2(-\mathbf{z}_1^{T}\boldsymbol{\alpha}_1, -\mathbf{z}_2^{T}\boldsymbol{\alpha}_2, \rho)]$$
$$\vdots$$
$$\begin{aligned} P(y_1 = j, y_2 = k|s = 1) \\ &= [\Phi(\mathbf{x}^{\mathbf{T}}\boldsymbol{\lambda})]\Big(\Phi_2(\delta_{j-1} - \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\alpha}_1, \gamma_{k-1} - \mathbf{z}_2^{\mathrm{T}}\boldsymbol{\alpha}_2, \rho) \\ &- \Phi_2(\delta_j - \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\alpha}, \gamma_k - \mathbf{z}_2^{\mathrm{T}}\boldsymbol{\alpha}_2, \rho)\Big) \end{aligned} \quad (2.9)$$
$$\vdots$$
$$\begin{aligned} P(y_1 = J, y_2 = K|s = 1 \\ &= \Phi(\mathbf{x}^{\mathbf{T}}\boldsymbol{\lambda}) \\ &+ [1 - \Phi(\mathbf{x}^{\mathbf{T}}\boldsymbol{\lambda})](1 - \Phi_2(\delta_{J-1} - \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\alpha}_1, \gamma_{K-1} - \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\alpha}_1, \rho)) \end{aligned}$$

3. RESULT

Parameter Estimation of ZIBOPR Model

Parameter estimation in the ZIBOPR model in this study uses the MLE method. The MLE method is an estimation method that is often used where this method maximizes the likelihood function. The ZIBOPR model is a probit regression model where the response variable for the ZIBOPR model follows a multinomial distribution. y for the ZIBOPR model follows a multinomial distribution. In the ZIBOPR model, there is an inflated variable which is a binary probit where ZIBOPR consists of a binary probit model and a bivariate ordinal probit model estimated together.

The distribution of the ZIBOPR model can be seen in the variable $\boldsymbol{Y} = [Y_{00} \quad Y_{01} \quad \dots \quad Y_{jk} \quad \dots \quad Y_{JK}]$where $Y_{jk} = 0{,}1$ for $j = 0, 1, 2, \dots, J$ and $k = 0, 1, 2, \dots, K$; $Y_{JK} = 1 - (Y_{00} + Y_{01} + Y_{02} + \dots + Y_{J,K-1})$. Joint probability $Y_1 = J, Y_2 = K$ is described in the following equation.

$$\begin{aligned} P(Y_1 = J, Y_2 = K) = 1 - P(Y_1 = 0, Y_2 = 0) + P(Y_1 = 0, Y_2 = 1) + \\ \dots + P(Y_1 = J - 1, Y_2 = K, (Y_1 = J, Y_2 = K - 1) \end{aligned}$$

So that the variables Y_1 and Y_2 which is the ZIBOPR model follows a multinomial distribution which is denoted as follows.

$$\boldsymbol{Y} \sim M(1; P_{00}, P_{01}, \dots, P_{J-1,K}, P_{J,K-1})$$

If a sample is taken as many as n, then the random sample is $y_1, y_2, \dots, y_n$ where $\boldsymbol{Y}_i = [\boldsymbol{Y}_{00i} \quad \boldsymbol{Y}_{01i} \quad \dots \quad Y_{jki} \quad \dots \quad Y_{(J-1,K)i}, Y_{(J,K-1)i}]$where $i = 1, 2, \dots, n$ and $\boldsymbol{Y}_i$ multinomial distribution which is denoted as follows.

$$\boldsymbol{Y}_i \sim M(n; P_{00i}, P_{01i}, \dots, P_{(J-1,K)i}, P_{(J,K-1)i})$$

In this study using four ordinal categories on two response variables, so it is $\boldsymbol{Y}_i$ denoted as follows.

$$\boldsymbol{Y}_i \sim M(1; P(Y_{1i} = 0, Y_{2i} = 0|\mathbf{x}_\mathrm{i}, \mathbf{z}_\mathrm{i}), P(Y_{1i} = 0, Y_{2i} = 1|\mathbf{x}_\mathrm{i}, \mathbf{z}_\mathrm{i}), \dots, P(Y_{1i} = 3, Y_{2i} = 3|\mathbf{x}_\mathrm{i}, \mathbf{z}_\mathrm{i})$$

Joint probability for Y_1 and Y_2 after taking a sample of n are described as follows.

$$P(y_{1i} = 0, y_{2i} = 0|s = 1) = \Phi(\mathbf{x}_\mathrm{i}^\mathbf{T}\boldsymbol{\lambda}) + [1 - \Phi(\mathbf{x}^\mathbf{T}\boldsymbol{\lambda})\Phi_2(-\mathbf{z}_{1\mathrm{i}}^{\boldsymbol{T}}\boldsymbol{\alpha}_1, -\mathbf{z}_{2\mathrm{i}}^{\boldsymbol{T}}\boldsymbol{\alpha}_\mathbf{2}, \rho)]$$

$$\vdots$$

$$\begin{aligned} P(y_1 = j, y_2 = k|s = 1) & \\ &= [\Phi(\mathbf{x}_\mathrm{i}^\mathbf{T}\boldsymbol{\lambda})]\Big(\Phi_2(\delta_{j-1} - \mathbf{z}_{1\mathrm{i}}^\mathrm{T}\boldsymbol{\alpha}_1, \gamma_{k-1} - \mathbf{z}_{2i}^\mathrm{T}\boldsymbol{\alpha}_2, \rho) \\ &\quad - \Phi_2(\delta_j - \mathbf{z}_{1\mathrm{i}}^\mathbf{T}\boldsymbol{\alpha}, \gamma_k - \mathbf{z}_{2i}^\mathrm{T}\boldsymbol{\alpha}_2, \rho)\Big) \end{aligned}$$

$$\vdots$$

$$P(y_1 = 3, y_2 = 3|s = 1) = [\Phi(\mathbf{x}_\mathrm{i}^\mathbf{T}\boldsymbol{\lambda})](1 - \Phi_2(\delta_{J-1} - \mathbf{z}_{1\mathrm{i}}^\mathrm{T}\boldsymbol{\alpha}_\mathbf{1}, \gamma_{K-1} - \mathbf{z}_{1\mathrm{i}}^\mathrm{T}\boldsymbol{\alpha}_1, \rho))$$

The estimated ZIBOPR model parameters are denoted by $\boldsymbol{\theta}$ where $\boldsymbol{\theta}$ the set of parameters $\boldsymbol{\theta} = \{\boldsymbol{\lambda}, \boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \boldsymbol{\rho}\}$, λ are the parameters of the binary probit in ZIBOPR, α_1 and α_2 are the parameters of bivariate ordinal probit in ZIBOPR, and ρ is the correlation coefficient between *errors* in bivariate ordinal probit in ZIBOPR. These parameters will be estimated using the MLE method and BHHH numerical iteration starting by finding the *ln* derivative of the probability of $\boldsymbol{\theta}$.

The derivative of *ln* joint probability with respect to $\boldsymbol{\lambda}$ which is then denoted $\boldsymbol{g}_i(\boldsymbol{\lambda})$ is obtained through Equation (3.1).

$$\frac{\partial \ln P(Y_1 = j, Y_2 = k)}{\partial \boldsymbol{\lambda}} = \frac{1}{P_{jki}(Y_{ji}, Y_{ki}|x_i, z_i)} \frac{\partial P_{jki}(Y_{ji}, Y_{ki}|x_i, z_i)}{\partial \boldsymbol{\lambda}} \tag{3.1}$$

The derivative of *ln* joint probability with respect to $\boldsymbol{\alpha}_1$ which is then denoted $\boldsymbol{g}_i(\boldsymbol{\alpha}_1)$ is obtained through Equation (3.2).

$$\frac{\partial \ln P(Y_1 = j, Y_2 = k)}{\partial \boldsymbol{\alpha}_1} = \frac{1}{P_{jk}(Y_{ji}, Y_{ki}|x_i, z_i)} \frac{\partial P_{jki}(Y_{ji}, Y_k i|x_i, z_i)}{\partial \boldsymbol{\alpha}_1} \tag{3.2}$$

The derivative of *ln* joint probability with respect to $\boldsymbol{\alpha_2}$ which is then denoted $\boldsymbol{g}_i(\boldsymbol{\alpha}_2)$ is obtained through Equation (3.3).

$$\frac{\partial \ln P(Y_1 = j, Y_2 = k)}{\partial \boldsymbol{\alpha}_2} = \frac{1}{P_{jki}(Y_{ji}, Y_{ki}|x_i, z_i)} \frac{\partial P_{jki}(Y_{ji}, Y_{ki}|x_i, z_i)}{\partial \boldsymbol{\alpha}_2} \tag{3.3}$$

The derivative of *ln* joint probability with respect to ρ which is then denoted $\boldsymbol{g}_i(\rho)$ is obtained through Equation (3.4).

$$\frac{\partial \ln P(Y_1 = j, Y_2 = k)}{\partial \rho} = \frac{1}{P_{jk}(Y_{ji}, Y_{ki}|x_i, z_i)} \frac{\partial P_{jki}(Y_{ji}, Y_{ki}|x_i, z_i)}{\partial \rho} \tag{3.4}$$

Then, construct the likelihood function for $\boldsymbol{\theta}$ is as follows.

$$L(\boldsymbol{\theta}) = \prod_{\boldsymbol{i=1}}^{\boldsymbol{n}} \left[\prod_{\boldsymbol{j,k=0}} [P(s=0) + P(s=1)P(r_1=0, r_2=0)]^{h_{ijk}} \times \prod_{j,k\neq 0} [P(s=1)P(r_1=j, r_2=k)]^{h_{ijk}} \right] \tag{3.5}$$

Then Equation (3.5) is transformed into the equation (3.6).

$$\ln L(\boldsymbol{\theta}) = \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \left[\sum_{\boldsymbol{j,k=0}} h_{ijk} \, [P(s=0) + P(s=1)P(r_1=0, r_2=0)] + \sum_{j,k\neq 0} h_{ijk} [P(s=1)P(r_1=j, r_2=k)] \right]$$

$$\begin{aligned} \ln L(\boldsymbol{\theta}) = \sum_{i=1}^{n} [d_{00i} \ln \left[\Phi(\mathbf{x}_\mathrm{i}^\mathrm{T}\boldsymbol{\lambda}) \left[\Phi_2\left(\delta_0 - \mathbf{z}_{1\mathrm{i}}^\mathrm{T}\boldsymbol{\alpha}_1, \gamma_0 - \mathbf{z}_{2i}^\mathrm{T}\boldsymbol{\alpha}_2, \rho\right)\right]\right] + \\ d_{i01} \ln \left[\left[\Phi(\mathbf{x}_\mathrm{i}^\mathbf{T}\boldsymbol{\lambda})\right] \left[\Phi_2\left(\delta_0 - \mathbf{z}_{1\mathrm{i}}^\mathrm{T}\boldsymbol{\alpha}_1, \gamma_1 - \mathbf{z}_{2i}^\mathrm{T}\boldsymbol{\alpha}_2, \rho\right) - \Phi_2\left(\delta_0 - \mathbf{z}_{1\mathrm{i}}^\mathbf{T}\boldsymbol{\alpha}, \gamma_0 - \mathbf{z}_{2i}^\mathrm{T}\boldsymbol{\alpha}_2, \rho\right)\right]\right] + \\ \ldots + d_{i33} \ln \left[\left[\Phi(\mathbf{x}_\mathrm{i}^\mathbf{T}\boldsymbol{\lambda})\right] \left[1 - \Phi_2\left(\delta_3 - \mathbf{z}_{1\mathrm{i}}^\mathrm{T}\boldsymbol{\alpha}_1, \gamma_3 - \mathbf{z}_{2i}^\mathrm{T}\boldsymbol{\alpha}_2, \rho\right)\right]\right] \end{aligned} \tag{3.6}$$

Then separate the parts ln and generalized by S which is described as follows.

$$\begin{aligned} \mathrm{S}_1 &= \sum_{i=1}^{n} d_{i00} \ln \left[\left[\Phi(\mathbf{x}_\mathrm{i}^\mathbf{T}\boldsymbol{\lambda})\right] + \left[1 - \Phi(\mathbf{x}_\mathrm{i}^\mathbf{T}\boldsymbol{\lambda})\right] \left[\Phi_2\left(\delta_0 - \mathbf{z}_{1\mathrm{i}}^\mathrm{T}\boldsymbol{\alpha}_1, \gamma_0 - \mathbf{z}_{2i}^\mathrm{T}\boldsymbol{\alpha}_2, \rho\right)\right]\right] \\ \mathrm{S}_2 &= \sum_{i=1}^{n} d_{i01} \ln \left[\left[\Phi(\mathbf{x}_\mathrm{i}^\mathbf{T}\boldsymbol{\lambda})\right] \left[\Phi_2\left(\delta_0 - \mathbf{z}_{1\mathrm{i}}^\mathrm{T}\boldsymbol{\alpha}_1, \gamma_1 - \mathbf{z}_{2i}^\mathrm{T}\boldsymbol{\alpha}_2, \rho\right) - \Phi_2\left(\delta_0 - \mathbf{z}_{1\mathrm{i}}^\mathbf{T}\boldsymbol{\alpha}, \gamma_0 - \mathbf{z}_{2i}^\mathrm{T}\boldsymbol{\alpha}_2, \rho\right)\right]\right] \\ &\vdots \\ \mathrm{S}_{16} &= \sum_{i=1}^{n} d_{i33} \ln \left[\left[\Phi(\mathbf{x}_\mathrm{i}^\mathbf{T}\boldsymbol{\lambda})\right] \left[1 - \Phi_2\left(\delta_3 - \mathbf{z}_{1\mathrm{i}}^\mathrm{T}\boldsymbol{\alpha}_1, \gamma_3 - \mathbf{z}_{2i}^\mathrm{T}\boldsymbol{\alpha}_2, \rho\right)\right]\right] \end{aligned} \tag{3.7}$$

Maximize the ln likelihood function in Equation (3.7) by lowering the equation with respect to the parameters $\boldsymbol{\theta} = \{\lambda, \alpha_1, \alpha_2, \rho\}$ and then equated to zero. To get the parameter estimate of $\boldsymbol{\lambda}$ is described in the following equation.

$$\begin{aligned} \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \boldsymbol{\lambda}} &= \frac{\partial}{\partial \boldsymbol{\lambda}} \sum_{j=0}^{J} \sum_{k=0}^{K} \sum_{i=1}^{n} d_{jki} \ln P_{jki} \left(Y_{ji}, Y_{ki} | x_i, z_i\right) \\ &= \sum_{j=0}^{J} \sum_{k=0}^{K} \sum_{i=1}^{n} d_{jki} \frac{1}{P_{jk}(Y_j, Y_k | x_i, z_i)} \frac{\partial P_{jki}(Y_{ji}, Y_{ki} | x_i, z_i)}{\partial \boldsymbol{\lambda}} \end{aligned} \tag{3.8}$$

Parameter estimator $\boldsymbol{\lambda}$ can be obtained by maximizing the ln likelihood function. Maximizing the *ln likelihood* function that has been separated in Equation (3.7) by lowering the equation against the parameter $\boldsymbol{\lambda}$. The derivative results are then summed up to obtain the parameter estimator $\boldsymbol{\lambda}$ which is described in the following equation.

$$\frac{\partial \ln L(\theta)}{\partial \boldsymbol{\lambda}} = \frac{\partial S_1}{\partial \boldsymbol{\lambda}} + \frac{\partial S_2}{\partial \boldsymbol{\lambda}} + \frac{\partial S_3}{\partial \boldsymbol{\lambda}} + \ \dots + \frac{\partial S_{16}}{\partial \boldsymbol{\lambda}} \tag{3.9}$$

Equation (3.9) is then equated to zero to obtain an estimator of the parameter $\boldsymbol{\lambda}$. After equating to zero, based on previous research, it turns out that the equation is not closed form. Based on this, the parameter estimation process $\boldsymbol{\lambda}$ is continued with the BHHH numerical iteration process.

Then to get the parameter estimation of $\boldsymbol{\alpha}_1$ is described in Equation (3.10)

$$\begin{aligned}\frac{\partial \ln L(\theta)}{\partial \boldsymbol{\alpha}_1} &= \frac{\partial}{\partial \boldsymbol{\alpha}_1} \sum_{j=0}^{J} \sum_{k=0}^{K} \sum_{i=1}^{n} d_{jki} \ln P_{jki}\,(Y_{ji}, Y_{ki}|x_i, z_i) \\ &= \sum_{j=0}^{J} \sum_{k=0}^{K} \sum_{i=1}^{n} d_{jki} \frac{1}{P_{jki}(Y_{ji}, Y_{ki}|x_i, z_i)} \frac{\partial P_{jki}(Y_{ji}, Y_{ki}|x_i, z_i)}{\partial \boldsymbol{\alpha}_1}\end{aligned} \tag{3.10}$$

In the same way, the parameter estimator is obtained $\boldsymbol{\alpha}_1$. Based on Equation (3.7), the derivative results are then summed up to obtain the parameter estimator $\boldsymbol{\alpha}_1$ which is described in the following equation.

$$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \boldsymbol{\alpha}_1} = \frac{\partial S_1}{\partial \boldsymbol{\alpha}_1} + \frac{\partial S_2}{\partial \boldsymbol{\alpha}_1} + \frac{\partial S_3}{\partial \boldsymbol{\alpha}_1} + \ \dots + \frac{\partial S_{16}}{\partial \boldsymbol{\alpha}_1} \tag{3.11}$$

Equation (3.11) is then equated to zero to obtain an estimator of the parameter $\boldsymbol{\alpha}_1$. After equating to zero, based on previous research, it turns out that the equation is not closed form. Based on this, the parameter estimation process $\boldsymbol{\alpha}_1$ is continued with the BHHH numerical iteration process.

Then to get the parameter estimation of $\boldsymbol{\alpha}_2$ is described in equation (3.12)

$$\begin{aligned}\frac{\partial \ln L(\theta)}{\partial \boldsymbol{\alpha}_2} &= \frac{\partial}{\partial \boldsymbol{\alpha}_2} \sum_{j=0}^{J} \sum_{k=0}^{K} \sum_{i=1}^{n} d_{jki} \ln P_{jki}\,(Y_{ji}, Y_{ki}|x_i, z_i) \\ &= \sum_{j=0}^{J} \sum_{k=0}^{K} \sum_{i=1}^{n} d_{jki} \frac{1}{P_{jki}(Y_{ji}, Y_{ki}|x_i, z_i)} \frac{\partial P_{jki}(Y_{ji}, Y_{ki}|x_i, z_i)}{\partial \boldsymbol{\alpha}_2}\end{aligned} \tag{3.12}$$

In the same way, the parameter estimator is obtained $\boldsymbol{\alpha}_2$**.** The derivative results are then summed up to obtain the parameter estimator $\boldsymbol{\alpha}_2$ which is described in the following equation.

$$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \boldsymbol{\alpha}_2} = \frac{\partial S_1}{\partial \boldsymbol{\alpha}_2} + \frac{\partial S_2}{\partial \boldsymbol{\alpha}_2} + \frac{\partial S_3}{\partial \boldsymbol{\alpha}_2} + \ \dots + \frac{\partial S_{16}}{\partial \boldsymbol{\alpha}_2} \tag{3.13}$$

Equation (3.13) is then equated to zero to obtain an estimator of the parameter $\boldsymbol{\alpha}_2$. After equating to zero, based on previous research, it turns out that the equation is not closed form. Based on this, the parameter estimation process $\boldsymbol{\alpha}_2$ is continued with the BHHH numerical iteration process.

Then to get the parameter estimation of ρ is described in the following equation.

$$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \rho} = \frac{\partial}{\partial \rho}\sum_{j=0}^{J}\sum_{k=0}^{K}\sum_{i=1}^{n} d_{jki} \ln P_{jki}\,(Y_{ji}, Y_k i|x_i, z_i)$$
$$= \sum_{j=0}^{J}\sum_{k=0}^{K}\sum_{i=1}^{n} d_{jki} \frac{1}{P_{jki}(Y_{ji}, Y_{ki}|x_i, z_i)} \frac{\partial P_{jk} i(Y_{ji}, Y_{ki}|x_i, z_i)}{\partial \rho} \tag{3.14}$$

In the same way, the parameter estimator is obtained ρ**.** The derivative results are then summed up to obtain the parameter estimator ρ which is described in the following equation.

$$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \rho} = \frac{\partial S_1}{\partial \rho} + \frac{\partial S_2}{\partial \rho} + \frac{\partial S_3}{\partial \rho} + \ldots + \frac{\partial S_{16}}{\partial \rho} \tag{3.15}$$

Equation (3.15) is then equated to zero to obtain an estimator of the parameter ρ. After equating to zero, based on previous research, it turns out that the equation is not closed form. Based on this, the parameter estimation process ρ is continued with the BHHH numerical iteration process.

Based on the MLE method previously described the estimation of parameters that are not *closed to* get a parameter estimator is obtained through numerical iteration using the BHHH method. $\boldsymbol{\theta} = \{\boldsymbol{\lambda}, \boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \rho\}$ obtained through numerical iteration using the BHHH method. The stages of the BHHH iteration procedure are as follows.

1. Initial initiation for parameters $\boldsymbol{\theta}^{(0)}$where $\boldsymbol{\theta}^{(0)} = [\lambda^{(0)} \quad \alpha_1^{(0)} \quad \alpha_2^{(0)} \quad \rho^{(0)}]$ initials are obtained from the estimated bivariate ordinal probit regression.
2. Evaluate the likelihood function and gradient obtained from the previous MLE estimation step.
3. Determine the gradient vector with the following equation.
$$\boldsymbol{g}(\widehat{\boldsymbol{\theta}}) = \left[\frac{\partial \ln L(.)}{\partial \boldsymbol{\lambda}} \quad \frac{\partial \ln L(.)}{\partial \boldsymbol{\alpha_1}} \quad \frac{\partial \ln L(.)}{\partial \boldsymbol{\alpha_1}} \quad \frac{\partial \ln L(.)}{\partial \boldsymbol{\rho}}\right]$$
4. Determine the Hessian matrix in equation (3.16)
$$\mathbf{H}(\boldsymbol{\theta}^{(l)}) = -\sum_{i=1}^{n} \boldsymbol{g}_i^T(\boldsymbol{\theta}^{(l)})\boldsymbol{g}_i(\boldsymbol{\theta}^{(l)}) \tag{3.16}$$

   where $\boldsymbol{g}_i(\boldsymbol{\theta}^{(l)})$ is the gradient vector of the *ln* Probability at each *i-th* sample of size $(p+q) \times 1$
$$\boldsymbol{g}_i(\boldsymbol{\theta}^{(l)}) = \frac{\partial \ln P(Y_{1i} = y_{1i}, Y_{2i} = y_{2i})}{\partial \boldsymbol{\theta}} \tag{3.17}$$

5. Perform parameter updating with equation (3.18).
$$\widehat{\boldsymbol{\theta}}^{(l+1)} = \widehat{\boldsymbol{\theta}}^{(l)} - \mathbf{H}^{-1}(\boldsymbol{\theta}^{(l)})\, \boldsymbol{g}(\widehat{\boldsymbol{\theta}}^{(l)}) \tag{3.18}$$

6. Repeat steps 2 through 4 and stop when converging i.e. $\left\|\boldsymbol{\theta}^{l+1}-\boldsymbol{\theta}^{l}\right\|<\varepsilon$ where ε is a very small tolerance value and $\varepsilon>0$.
7. Iteration stops when $l=M$ then we get $\widehat{\boldsymbol{\theta}}=\widehat{\boldsymbol{\theta}}^{(M)}$.

Marginal Effect

The marginal effect in ZIOP regression is defined as the change in probability in the response variable when the predictor variable changes. The marginal effect in ZIBOPR regression for J and K categories is by reducing the ZIBOPR regression model to its predictor variables. The marginal effect equation for the ZIBOPR regression model is described in equation (3.19) [7].

$$\begin{aligned}
&\mathrm{ME}(P(Y_1=0,Y_2=0) \\
&\qquad=\frac{\partial P(Y_1=0,Y_2=0)}{\partial \mathbf{x}}+\frac{\partial P(Y_1=0,Y_2=0)}{\partial \mathbf{z}_1}+\frac{\partial P(Y_1=0,Y_2=0)}{\partial \mathbf{z}_2} \\
&\vdots \\
&\mathrm{ME}(P(Y_1=J,Y_2=K) \\
&\qquad=\frac{\partial P(Y_1=J,Y_2=K)}{\partial \mathbf{x}}+\frac{\partial P(Y_1=J,Y_2=K)}{\partial \mathbf{z}_1}+\frac{\partial P(Y_1=0,Y_2=0)}{\partial \mathbf{z}_2}
\end{aligned} \tag{3.19}$$

4. CONCLUSION

Based on the previous discussion, it can be concluded that the ZIBOPR model can be applied to regression models with two multinomial distributed response variables. The ZIBOPR model is a development of the ordinal probit regression model where both response variables are zero-inflated. A zero-inflated condition occurs when one category in both responses is overrepresented compared to the other category. In this condition, estimation using bivariate ordinal probit regression will be biased because this method is unable to handle zero-inflated data. The ZIBOPR model is a combination of binary probit regression and bivariate ordinal probit regression models. Parameters in the ZIBOPR model can be estimated using the MLE method. The equation from MLE estimation process is not closed form, so the estimation process is continued with the BHHH iteration.

REFERENCES

[1] Bagozzi, B. E., Hill, D. W., Moore, W. H., & Mukherjee, B., 2015. Modeling Two Types of Peace: The Zero-inflated Ordered Probit (ZIOP) Model in Conflict Research. *Journal of Conflict Resolution*, Vol. 59, No. 4, 728-752.

[2] Daganzo, C., 1979. *Multinomial Probit. The Theory and Its Application to demand forecasting*. Academic Press. California.

[3] Fu, C., Chen., C., 2025. Studying Older Adults' Willingness To Relocate And Change Transport Mode With The Random Bivariate Generalized Ordered Probit. *Transportation Research Part A*, Vol. 191, No. 104322, 1 – 22.

[4] Ghanbarnejad, A., Homayuni, A., Hosseini, Z., & Madani, A., 2022. Smoking Behavior Among Students: Using Health Belief Model and Zero-Inflated Ordered Probit Model. *Tobacco and Health*, Vol. 1, No. 2, 74 – 82.

[5] Gkartzonikas, C., Dimitriou, L., 2023. Shared Micro-Mobility Services for University Communities: A Multivariate Ordered Probit Approach. *Transportation Research Record*, Vol. 2677, No. 11, 148 – 168.

[6] Greene, W. H., & Hansher, D. A., 2010. *Modelling Ordered Choices: A Primer*. Cambridge University Press. Cambridge.

[7] Gurmu, M., & Dagne, G. A., 2012. Bayesian Approach to Zero-inflated Bivariate Ordered Probit Regression Model, with an Application to Tobacco Use. *J Probab Stat*, Vol. 2012, No. 617678, 1–26.

[8] Jiang, X., Huang, B., Zaretzki, R. L., Richards, S., Yan, X., & Zhang, H., 2013. Investigating the Influence of Curbs on Single-Vehicle Crash Injury Severity Utilizing Zero-Inflated Ordered Probit Models. *Accid Anal Prev*, Vol. 57, No. 3, 55–66.

[9] Mai, A. T., Bastin, F., Toulouse, M., 2014. On Optimization Algorithms for Maximum Likelihood Estimation. *CIRRELT: Interuniversity Research Centre on Enterprise Networks, Logistics and Transportation.*

[10] Nazillah, A. H., Melanian, S., 2022. Analisis Regresi Ordinal Model Probit Terhadap Faktor yang Mempengaruhi Indeks Pembangunan Manusia di Jawa Timur. *Jurnal Kesehatan Masyrakat*, Vol. 13, No. 2, 267 – 278.

[11] Rejeki, N., Ratnasari, V., & Ahsan, M., 2024. Modelling of Poor Household in East Kalimantan Using Zero Inflated Ordered Probit (ZIOP) Approach. *Procedia Computer Science*, Vol. 234, No. 1, 278–285.

[12] Scott, J.L., 1997. *Regression Model for Categorical and Limited Dependent Variable*. Sage. California.

[13] Wu, J., Fan, W., & Wencheng, W., 2020. A Zero-Inflated Ordered Probit Model to Analyze Hazmat Truck Driver's Violation Behavior and Associated Risk Factors. *Special Section on Big Data Technology and Applications in Intelligent Transportation*, Vol. 8, No. 1, 110974–110985.

[14] Yakubu, M.A., Aidoo, E. N., Ampofo, R. T., & Ackaah, W., 2024. Bivariate ordered probit modelling of motorcycle riders and pillion passengers' injury severities relationship and associated risk factors. *International Journal of Injury Control and Safety Promotion*, Vol. 31, No. 3, 499–507.

[15] Yamamoto, T., & Shankar, V. N., 2004. Bivariate Ordered-Response Probit Model of Driver's and Passenger's Injury Severities in Collisions with Fixed Objects. *Accid Anal Prev*, Vol. 36, No. 5, 869–876.

[16] Zhiheng, F., Yuan, R., Xiang, Q., 2024. An Exploratory Investigation into The Influence of Risk Factors On Driver Injury Severity In Angle Crashes: A Random Parameter Bivariate Ordered Probit Model Approach. *Engineering & Technology Journals*, Vol. 25, No. 1, 70–77.