JURNAL MATEMATIKA, STATISTIKA DAN KOMPUTASI e–ISSN: 2614–8811
Published by Departement of Mathematics, Hasanuddin University, Indonesia
https://journal.unhas.ac.id/index.php/jmsk/index
Vol. 22, No. 2, January 2026, pp. 315-321
DOI: 10.20956/j.v22i2.48175

A Study of (R, S)-Bimodules Homomorphisms

Dian Ariesta Yuwaningsih[1,*], Indah Emilia Wijayanti[2], Budi Surodjo[3]

[1,*]*Department of Mathematics Education, Universitas Ahmad Dahlan, Yogyakarta, Indonesia*
[2,3]*Department of Mathematics, Universitas Gadjah Mada, Yogyakarta, Indonesia*
***Email**: [1]dian.ariesta@pmat.uad.ac.id, [2]ind_wijayanti@ugm.ac.id, [3]surodjo_b@ugm.ac.id*
[*]*Corresponding Author*

Abstract

This paper discusses the generalization of the fundamental theorems of R-module homomorphisms to the structure of (R, S)-bimodules, where R and S are rings with identity. The study begins with a review of the definitions, properties, and types of (R, S)-bimodule homomorphisms. Subsequently, three fundamental theorems of R-module homomorphisms are generalized to the (R, S)-bimodule setting. The results show that the fundamental structures and relationships in module theory can be naturally extended to bimodules by considering the actions of two rings that are compatible with the bimodule operations. This generalization provides a broader framework for studying algebraic structures involving two interacting ring actions.
Keywords: homomorphism, isomorphism, epimorphisms, (R, S)-bimodule

1. Introduction

Throughout this paper, R and S are rings with identity, and M is an Abelian additive group, unless stated otherwise. In ring theory, there exists a function that preserves both addition and multiplication, called a ring homomorphism. Let R and S be rings with idenitity. Based on [1], a function $f: R \to S$ is called a ring homomorphism if, for each $a, b \in R$, it satisfies $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$. Moreover, Malik [1] also discussed the types and properties of ring homomorphisms. The types of rings homomorphisms include ring monomorphisms, ring epimorphisms, and ring isomorphisms. Furthermore, discussions related to the ring homomorphisms are often concluded with the fundamental theorem of ring isomorphisms. A more detailed treatment of ring homomorphisms can also be found in [2-5].

In its development, all theories in ring structures have been extended to module structures, including theories related to ring homomorphisms. Let M and N be R-modules. Based on [3], a function $f: M \to N$ is called an R-module homomorphism if for each $m, n \in M$ and $r \in R$, it satisfies $f(m + n) = f(m) + f(n)$ and $f(rm) = rf(m)$. Similar to the case of rings, the types of R-module homomorphisms include R-module monomorphisms, R-module epimorphisms, and R-module isomorphisms. The properties of R-module homomorphism are generalizations of the properties of ring homomorphisms. Furthermore, the study of R-module homomorphisms is often concluded with the fundamental theorem of R-module isomorphisms. The notion of R-module homomorphism has also been extensively studied by various authors, such as in [3-5].

As algebraic theory developed, the concept of a bimodule was introduced by Adkins [3] as an extension of a module equipped with two compatible ring actions, one from the left by R and one from the right by S. In recent years, several researchers in the field of algebra have increasingly focused on the study of (R, S)-bimodule structures, as discussed in papers [6–9]. In their studies, they employed the concept of bimodule homomorphisms, but the fundamental theorems of bimodule homomorphisms were not utilized. The concept of the types of modules homomorphisms and the fundamental theorems of module isomorphisms has not yet been explicitly developed for bimodule structures. Adkins [3] discusses only the definition of bimodule homomorphisms without providing an in-depth treatment of this topic. However, Yuwaningsih *et al.* [10] have developed the types of homomorphisms and the fundamental theorems of isomorphisms for the (R, S)-module structure. This (R, S)-module structure is a generalization of the bimodule structure introduced by Khumprapussorn *et al.* [11].

This paper aims to generalize the fundamental theorems of module homomorphisms to the structure of (R, S)-bimodules. Before presenting this generalization, the definitions, properties, and types of bimodule homomorphisms are reviewed. The results are expected to provide a broader framework for understanding the structural relationships between modules and bimodules, thereby contributing to the further development of algebraic theory.

2. Some Properties of (R, S)-Bimodule Homomorphisms

This section is started by the definition of (R, S)-bimodule homomorphisms according to Adkins [3].

Definisi 2.1. [3] Let M_1 and M_2 be (R, S)-bimodules. A function $f: M_1 \to M_2$ is called an (R, S)-bimodule homomorphism if f is a left R-module homomorfisms and a right S-module homomorphism.

Based on the above definition, a function $f: M_1 \to M_2$ is called an (R, S)-bimodule homomorphism if for each $m, n \in M_1$, $r \in R$, and $s \in S$ satisfy $f(m + n) = f(m) + f(n)$, $f(rm) = rf(m)$, and $f(ms) = f(m)s$.

Before we consider several examples of (R, S)-bimodule homomorphisms, we show some basic properties of (R, S)-bimodule homomorphisms.

Proposisi 2.2. Let $f: M_1 \to M_2$ be an (R, S)-bimodule homomorphism. Then,

a) f preserves the neutral element, that is $f\left(0_{M_1}\right) = 0_{M_2}$.

b) f preserves the inverse element of each element in M_1, that is $f(-m) = -f(m)$ for all $m \in M_1$.

c) If H is an (R, S)-bisubmodule of M_1, then $f(H) = \{f(h) \mid h \in H\}$ is an (R, S)-bisubmodule of M_2.

d) If N is an (R, S)-bisubmodule of M_2, then $f^{-1}(N) = \{m \in M_1 \mid f(m) \in N\}$ is an (R, S)-bisubmodule of M_1.

Proof. Let $f: M_1 \to M_2$ be an (R, S)-bimodule homomorphism.

a) Since f is an (R,S)-bimodule homomorphism, $f(0_{M_1}) + f(0_{M_1}) = f(0_{M_1} + 0_{M_1}) = f(0_{M_1}) = f(0_{M_1}) + 0_{M_2}$. This implies that $f(0_{M_1}) = 0_{M_2}$.

b) Let $m \in M_1$. Then, $f(m) + f(-m) = f(m + (-m)) = f(0_{M_1}) = 0_{M_2}$. Similarly, $f(-m) + f(m) = f(-m + m) = f(0_{M_1}) = 0_{M_2}$. Since $f(m)$ has a unique invers, $f(-m) = -f(m)$.

c) Let H be an (R,S)-bisubmodule of M_1. Then, $0_{M_1} \in H$ and by (a), $f(0_{M_1}) = 0_{M_2}$. Thus, $0_{M_2} = f(0_{M_1}) \in f(H)$ and so $f(H) \neq \emptyset$. Let any $x, y \in f(H)$ where $x = f(h_1)$ and $y = f(h_2)$. Since f is an (R,S)-bimodule homomorphism, then $x - y = f(h_1) - f(h_2) = f(h_1 - h_2) \in f(H)$. Furthermore, let any $r \in R$ and $s \in S$. Since f is an (R,S)-bimodule homomorphism, then $rx = rf(h_1) = f(rh_1) \in f(H)$ dan $xs = f(h_1)s = f(h_1 s) \in f(H)$. Hence, $f(H)$ is an (R,S)-submodule of M_2.

d) Let N be an (R,S)-bisubmodule of M_2. It's clear that $f^{-1}(N) \neq \emptyset$ because $0_M \in f^{-1}(N)$. Let any $x, y \in f^{-1}(N)$. Then $f(x) \in N$ and $f(y) \in N$. Since f is an (R,S)-bimodule homomorphism, then $f(x - y) = f(x) - f(y)$. Since $f(x) \in N$ and $f(y) \in N$, then $f(x) - f(y) \in N$, so $x - y \in f^{-1}(N)$. Furthermore, let any $r \in R$ dan $s \in S$. Since f is an (R,S)-bimodule homomorphism then $f(rx) = rf(x)$ dan $f(xs) = f(x)s$. Since $f(x) \in N$ and N is an (R,S)-submodule then $rf(x) \in N$ and $f(x)s \in N$, so $rx \in f^{-1}(N)$ dan $xs \in f^{-1}(N)$. Hence, $f^{-1}(N)$ is an (R,S)-submodule of M_1. □

Definition 2.3. Let $f: M_1 \to M_2$ be an (R,S)-bimodule homomorphism. The kernel of f, written $Ker(f)$, is defined to be the set $Ker(f) = \{m \in M_1 \mid f(m_1) = 0_{M_2}\}$.

Definition 2.4. Let $f: M_1 \to M_2$ be an (R,S)-bimodule homomorphism. The image of f, written $Im(f)$, is defined to be the set $Im(f) = \{y \in M_2 \mid (\exists\, m \in M_1)\; f(m) = y\}$.

Now, we present several examples of (R,S)-bimodule homomorphisms.

Example 2.5. Let M and N be (R,S)-bimodules and a zero function $\theta: M \to N$ with $\theta(m) = 0_N$ for all $m \in M$. Let any $x, y \in M$, $r \in R$, and $s \in S$. We obtain $\theta(x + y) = 0_N = 0_N + 0_N = \theta(x) + \theta(y)$, $\theta(rx) = 0_N = r0_N = r\theta(x)$, and $\theta(xs) = 0_N = 0_N s = \theta(x)s$. Thus, the zero function θ is an (R,S)-bimodule homomorphism. Moreover, clearly that $Ker(\theta) = M$ and $Im(\theta) = \{0_N\}$.

Example 2.6. Let $\mathbb{Z}$ and $\mathbb{Z}[x]$ be $(\mathbb{Z},\mathbb{Z})$-bimodules. A function $g: \mathbb{Z} \to \mathbb{Z}[x]$ with $g(a) = ax^2$ for each $a \in \mathbb{Z}$ is an $(\mathbb{Z},\mathbb{Z})$-bimodule homomorphism since for all $r, s, a, b \in \mathbb{Z}$ satisfy:

i) $g(a + b) = (a + b)x^2 = ax^2 + bx^2 = g(a) + g(b)$,

ii) $g(ra) = (ra)x^2 = r(ax^2) = rg(a)$,

iii) $g(as) = (as)x^2 = (ax^2)s = g(a)s$.

Furthermore, we obtain $Ker(g) = \{0\}$ and $Im(g) = \{mx^2 \in \mathbb{Z}[x] \mid m \in \mathbb{Z}\}$.

Example 2.7. Let $\mathbb{Z}$ be an $(2\mathbb{Z}, 2\mathbb{Z})$-bimodule. A function $f: \mathbb{Z} \to \mathbb{Z}$ with $f(a) = 5a$ for all $a \in \mathbb{Z}$ is an $(2\mathbb{Z}, 2\mathbb{Z})$-bimodule homomorphism. Let any $a, b \in \mathbb{Z}$ and $r, s \in 2\mathbb{Z}$, we obtain $f(a + b) = 5(a + b) = 5a + 5b = f(a) + f(b)$, $f(ra) = 5(ra) = r(5a) = rf(a)$, dan $f(as) = 5(as) = (5a)s = f(a)s$. Thus, f is an $(2\mathbb{Z}, 2\mathbb{Z})$-bimodule homomorphism. Moreover, we obtain $Ker(f) = \{0\}$ and $Im(f) = 5\mathbb{Z}$.

The following is given some properties showed that the kernel and the image of any (R,S)-bimodule homomorphism is an (R,S)-bisubmodule.

Proposition 2.8. Let $f: M_1 \to M_2$ be an (R,S)-bimodule homomorphism. Then the kernel of f is an (R,S)-bisubmodule of M_1.

Proof. Obviously that $Ker(f) \subseteq M_1$. Since $f(0_{M_1}) = 0_{M_2}$, then $0_{M_1} \in Ker(f)$ so we get $Ker(f) \neq \emptyset$. Let any $x, y \in Ker(f)$, then $f(x) = 0_{M_2}$ and $f(y) = 0_{M_2}$. Since f is an (R,S)-bimodule homomorphism, then $f(x-y) = f(x) - f(y) = 0_{M_2} - 0_{M_2} = 0_{M_2}$. Thus, $x - y \in Ker(f)$. Furthermore, let any $r \in R$ and $s \in S$. Since f is an (R,S)-bimodule homomorphism, then $f(rx) = rf(x) = r0_{M_2} = 0_{M_2}$ and $f(xs) = f(x)s = 0_{M_2}s = 0_{M_2}$. So, $rx \in Ker(f)$ and $xs \in Ker(f)$. Thus, $Ker(f)$ is an (R,S)-bisubmodule of M_1. □

Proposition 2.9. Let $f: M_1 \to M_2$ be an (R,S)-bimodule homomorphism. Then the image of f is an (R,S)-bisubmodule of M_2.

Proof. Obviously that $Im(f) \subseteq M_2$. Since $f(0_{M_1}) = 0_{M_2} \in Im(f)$, then $Im(f) \neq \emptyset$. Let any $x, y \in Im(f)$, then $x = f(a)$ and $y = f(b)$ where $a, b \in M_1$. Since f is an (R,S)-bimodule homomorphism, then $x - y = f(a) - f(b) = f(a-b)$. Since $a, b \in M_1$, then $a - b \in M_1$ so we get $x - y \in Im(f)$. Furthermore, let any $r \in R$ and $s \in S$. Since f is an (R,S)-bimodule homomorphism, then $rx = rf(a) = f(ra)$ and $xs = f(a)s = f(as)$. Since $a \in M_1$, then $ra, as \in M_1$, so we obtain $rx \in Im(f)$ and $xs \in Im(f)$. Thus, $Im(f)$ is an (R,S)-bisubmodule of M_2. □

Next, let M be an (R,S)-bimodule and N an (R,S)-bisubmodule of M. Since N is a subgroup of the additive Abelian group M, we can form the factor group M/N. Because M is an Abelian group, it is clear that M/N is also Abelian. The scalar multiplication on the Abelian group M/N is defined as follows:

$$r(m + N) = rm + N$$

and

$$(m + N)s = ms + N$$

for every $m + N \in M/N$, $r \in R$, and $s \in S$. We can show that both scalar multiplication opertaions are well-defined. If $a + N, b + N \in M/N$ with $a + N = b + N$, then $a - b \in N$. Consequently, for any $r \in R$, we have $ra - rb = r(a - b) \in N$, so $ra + N = rb + N$. Furthermore, for any $s \in S$, we obtain $as - bs = (a - b)s \in N$, hence $as + N = bs + N$. Therefore, both scalar multiplication operations are well-defined. The (R,S)-bimodule M/N is then called the factor (R,S)-bimodule of M with respect to the (R,S)-bisubmodule N.

3. The Fundamental Theorem of (R,S)-Bimodule Homomorphisms

In this section, we present the types of (R,S)-bimodule homomorphisms, several examples related to types of (R,S)-bimodule homomorphisms, the natural (R,S)-bimodule homomorphisms, and the fundamental theorem of (R,S)-bimodule isomorphisms. On module theory, there are some types of R-module homomorphism such as R-module epimorphism, R-module monomorphism, and R-module isomorphism. These types of R-module homomorphism has been studied in [3]. Now, below we present the types of (R,S)-bimodule homomorphism as follows.

Definisi 3.1. Let $f: M_1 \to M_2$ be an (R,S)-bimodule homomorphism.

a) A function f is called an (R,S)-bimodule monomorphism if f is injective.

b) A function f is called an (R,S)-bimodule epimorphism if f is surjective.

c) A function f is called an (R,S)-bimodule isomorphism if f is bijective.

Moreover, the (R,S)-bimodule M_1 and (R,S)-bimodule M_2 is called isomorphic if there exist an (R,S)-bimodule isomorphism from M_1 to M_2, denoted by $M_1 \cong M_2$.

Example 3.2. Let $\mathbb{Z}$ be an $(2\mathbb{Z},2\mathbb{Z})$-bimodule. A function $f:\mathbb{Z} \to \mathbb{Z}$ with $f(a) = -a$ for all $a \in \mathbb{Z}$ is an $(2\mathbb{Z},2\mathbb{Z})$-bimodule isomorphism. Let any $a,b \in \mathbb{Z}$ and $r,s \in 2\mathbb{Z}$, we obtain $f(a+b) = -(a+b) = -a-b = -a+(-b) = f(a)+f(b)$, $f(ra) = -(ra) = r(-a) = rf(a)$, and $f(as) = -(as) = (-a)s = f(a)s$. Thus, f is an (R,S)-bimodule homomorphism. Furthermore, if we asume that $f(a) = f(b)$, then $-a = -b$, so $a = b$. Thus, f is injective. Next, for each $a \in \mathbb{Z}$ here exist $-a \in \mathbb{Z}$ such that $a = -(-a) = f(-a)$. Thus, f is surjective. Hence, f is an $(2\mathbb{Z},2\mathbb{Z})$-bimodule isomorphism.

The following proposition gives a necessary and sufficient condition for a monomorphism to be an injective mapping in terms of its kernel.

Proposition 3.3. Let $f: M_1 \to M_2$ be an (R,S)-bimodule homomorphism. Then f is injective if and only if $Ker(f) = 0_{M_1}$.

Proof. Let any $a \in Ker(f)$. Then $f(a) = 0_{M_2} = f(0_{M_1})$. Since f is injective, we must have $a = 0_{M_1}$. Hence, $Ker(f) = \{0_{M_1}\}$. Conversely, let any $a,b \in M$ and we may assume $f(a) = f(b)$. Then $f(a-b) = f(a)-f(b) = f(a)-f(a) = 0_{M_2}$. Thus, we obtain $a-b \in Ker(f)$. Since $Ker(f) = \{0_{M_1}\}$, then $a-b = 0_{M_1}$ i.e. $a = b$. Thus, f is injective. □

According to Proposition 2.8, we know that if $f: M_1 \to M_2$ is an (R,S)-bimodule homomorphism, then $Ker(f)$ is an (R,S)-bisubmodule of M. In the following proposition, we show that every (R,S)-bisubmodule N of M induces an (R,S)-bimodule homomorphism g of M onto the factor (R,S)-bimodule M/N such that $Ker(g) = N$.

Proposition 3.4. Let M be an (R,S)-bimodule and N is an (R,S)-bisubmodule of M. Define the function g from M onto the factor (R,S)-bimodule M/N by $g(a) = a+N$ for all $a \in M$. Then, g is an (R,S)-bimodule homomorphism of M onto M/N and $Ker(g) = N$. Moreover, the (R,S)-bimodule homomorphism g is called the natural (R,S)-bimodule homomorphism of M onto M/N.

Proof. From the definition of g, it follows that g is a function from M onto M/N. To show g is homomorphism, let $a,b \in M$ $r \in R$ and $s \in S$. Then we obtain $g(a+b) = (a+b)+N = (a+N)+(b+N) = g(a)+g(b)$, $g(ra) = (ra)+N = r(a+N) = rg(a)$, and $g(as) = (as)+N = (a+N)s = g(a)s$. Hence, g is an (R,S)-bimodule homomorphism of M onto M/N. Next, we will show that $Ker(g) = N$. Now $a \in Ker(g)$ if and only if $g(a) = 0_M + N$ if and only if $a+N = 0_M + N$ if and only if $a \in N$. Thus, $Ker(g) = N$. □

Now, we give several properties that show us the relationship between (R,S)-bimodule homomorphisms, the kernel, the image, and the factor (R,S)-bimodule. This properties is known as the fundamentals theorem of (R,S)-bimodule homomorphism.

Theorem 3.5. (The First (R,S)-Bimodule Isomorphism Theorem) Let M_1 and M_2 be (R,S)-bimodules. Let $f: M_1 \to M_2$ be an (R,S)-bimodule homomorphism, then $M_1/Ker(f) \cong Im(f)$.

Proof. Let $K = Ker(f)$ and define $\theta: M_1/K \to Im(f)$ by $\theta(m+K) = f(m)$ for all $m+K \in M_1/K$. Let any $a+K, b+K \in M_1/K$. Now $a+K = b+K$ if and only if $a-b \in K$ if and only if $f(a-b) = 0_{M_2}$ if and only if $f(a) = f(b)$ if and only if $\theta(a+K) = \theta(b+K)$. Thus, θ is injective. Let any $y \in Im(f)$. Then, then $y = f(a)$ for some $a \in M_1$. Therefore, $\theta(a+K) =$

$f(a) = y$. This show that θ is surjective. Finally, let any $a + K, b + K \in M_1/K$, $r \in R$, and $s \in S$. Then,

$$\theta((a + K) + (b + K)) = \theta((a + b) + K) = f(a + b) = f(a) + f(b) = \theta(a + K) + \theta(b + K),$$

$$\theta(r(a + K)) = \theta((ra) + K) = f(ra) = rf(a) = r\theta(a + K),$$

and

$$\theta((a + K)s) = \theta((as) + K) = f(as) = f(a)s = r\theta(a + K).$$

Thus, θ is an (R, S)-bimodule homomorphism. Consequently, $M_1/Ker(f) \cong Im(f)$. □

If the function $f: M_1 \to M_2$ on the First (R, S)-Bimodule Isomorphism Theorem is an epimorphism, then we obtain $M_1/Ker(f) \cong M_2$.

Before proceeding to the next (R, S)-bimodule isomorphism theorem, note that the sum and intersection of two (R, S)-bisubmodules also form an (R, S)-bisubmodule.

Proposisi 3.6. Let M be an (R, S)-bimodule and let N and P be (R, S)-bisubmodules of M. Then,

a) $N + P$ is an (R, S)-bisubmodule of M.

b) $N \cap P$ is an (R, S)-bisubmodule of M.

Proof. Let N and P be (R, S)-bisubmodules of M.

a) It is clear that $N + P \neq \emptyset$ since $0_M = 0_M + 0_M \in N + P$. Let any $a, b \in N + P$ with $a = n_1 + p_1$ and $b = n_2 + p_2$. We obtain $a - b = (n_1 + p_1) - (n_2 + p_2) = (n_1 - n_2) + (p_1 - p_2) \in N + P$. Moreover, for any $r \in R$ and $s \in S$, we have $ra = r(n_1 + p_1) = rn_1 + rp_1 \in N + P$ and $as = (n_1 + p_1)s = n_1s + p_1s \in N + P$. Thus, $N + P$ is an (R, S)-bisubmodule of M.

b) It is clear that $N + P \neq \emptyset$ since $0_M \in N \cap P$. Let any $r \in R$, $s \in S$, and $a, b \in N \cap P$. Then $a, b \in N$ and $a, b \in P$. Since N and P are (R, S)-bisubmodules of M, we obtain $a - b \in N$ and $a - b \in P$, so $a - b \in N \cap P$. Moreover, we have $ra, as \in N$ and $ra, as \in P$. Hence $ra \in N \cap P$ and $as \in N \cap P$. Thus, $N \cap P$ is an (R, S)-bisubmodule of M. □

Subsequently, we present the second and third (R, S)-bimodule isomorphism theorems.

Theorem 3.6. (The Second (R, S)-Bimodule Isomorphism Theorem) Let M be an (R, S)-bimodule. If N and P be (R, S)-bisubmodules of M, then $(N + P)/P \cong N/(N \cap P)$.

Proof. Let $\pi: M \to M/P$ be the natural (R, S)-bimodule homomorphism and π_0 be the restriction of π to N. Then π_0 is an (R, S)-bimodule homomorphism with $Ker(\pi_0) = N \cap P$ and $Im(\pi) = (N + P)/P$. The result then follows from the First (R, S)-Bimodule Isomorphism Theorem. □

Theorem 3.7. (The Third (R, S)-Bimodule Isomorphism Theorem). Let M be an (R, S)-bimodule. If N and P be (R, S)-bisubmodules of M with $P \subseteq N$, then $M/N \cong (M/P)/(N/P)$.

Proof. Define $f: M/P \to M/N$ by $f(m + P) = m + N$. This is well defined (R, S)-bimodule homomorphism and $Ker(f) = \{m + P \mid m + N = N\} = \{m + P \mid m \in N\} = N/P$. The result then follows from the First (R, S)-Bimodule Isomorphism Theorem. □

4. Conclusion

In this paper, we have discussed the (R, S)-bimodule homomorphism, their properties, their types, and the fundamentals theorem of (R, S)-bimodule isomorphisms. The results show that the definitions and the properties of (R, S)-bimodule homomorphisms can be generalized from those of module homomorphisms.

Acknowledgement

The author would like to express their gratitude to the reviewers for their valuable comments and suggestions.

References

[1] Malik, D.S., Mordeson, J.N., and Sen, M.K., 1997. *Fundamentals of Abstract Algebra*, McGraw-Hill, New York.
[2] Fraleigh, J.B., 2000. *A First Course in Abstract Algebra*. Addison-Wesley Publishing Company, USA.
[3] Adkins, W.A., 1992. *Algebra* "*An Approach via Module Theory*". Springer-Verlag New York, Inc., USA.
[4] Dummit, D.S. and Foote, R.M., 2004. *Abstract Algebra*. John Wiley and Son, Inc., USA.
[5] Wijayanti I E, Wahyuni S, Yuwaningsih, D.A., and Hartanto, A.D., 2016. *Teori Ring dan Modul*. Gama Press UGM, Yogyakarta.
[6] Liu, J., Bai, C., and Sheng, Y., 2019. Compatible $\mathcal{O}$-operators on bimodules over associative algebras. *Journal of Algebra*, **532**(2019), 80-118.
[7] Javanshiri, H. and Nemati, M., 2018. Amalgamated duplication of the Banach algebra $\mathcal{A}$ along a $\mathcal{A}$-bimodule $\mathcal{A}$. *Journal of Algebra and Its Applications*, **17**(9), 1850169.
[8] Yuwaningsih, D.A., Wijayanti, I.E, and Surodjo, B., 2023. On Left r-Clean Bimodules. *Journal of Algebra and Related Topics*, **11**(2), 1-19.
[9] Agusfrianto, F.A., Fitriani, and Mahatma, Y., 2024. On Rough Bimodules. *AIP Conference Proceedings*, **2970**(03002).
[10] Yuwaningsih, D.A., Wijayanti, I.E., and Prasetyo, P.W., 2019. On (R, S)-Module Homomorphisms. *Journal of Physics: Conference Series*, **1188**, 012114.
[11] Khumprapussorn, T., Pianskool, S., and Hall, M., 2012. (R, S)-Modules and their Fully and Jointly Prime Submodules, *International Mathematical Forum*. **7**(33), 1631-1643.