

Hyper-Wiener and Harary Indices of The Line Graph of Zero Divisor Graph For Integer Modulo Ring

Fariz Maulana^{1*}, I Gede Adhitya Wisnu Wardhana²

^{1,2} Department of Mathematics, University of Mataram

Email: fariz.maulana@unram.ac.id, adhitya.wardhana@unram.ac.id

*Corresponding author

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Abstract

This paper studies the line graph of the zero divisor graph of the ring \mathbb{Z}_p^n , where p is a prime and n is a natural number. The main objective is to compute closed formulas for the Hyper-Wiener and Harary indices of the line graph $L(\Gamma\mathbb{Z}_p^n)$ and to analyze their relationship with the first Zagreb index of the original graph $\Gamma\mathbb{Z}_p^n$. The results establish connections between structural properties of the zero divisor graph and those of its corresponding line graph, offering a deeper understanding of interactions among degree-based and distance-based topological indices.

Keywords: Line Graph, Zero Divisor Graph, Integer modulo ring, Hyper Wiener index, Harary index, First Zagreb index

1. INTRODUCTION AND PRELIMINARIES

Topological indices play an important role in chemical documentation, isomer discrimination, investigating structure–property relationships, and various other applications [6]. Since the pioneering work of Wiener, numerous types of topological indices have been introduced and extensively studied, including the Wiener index [23], Zagreb indices [5,7], Harary index [24], and more recent indices such as the Sombor index [16,18]. These indices provide quantitative tools to capture structural information of graphs and establish meaningful relationships between graph invariants and underlying algebraic or chemical properties.

A zero divisor graph provides a fundamental graphical representation of an algebraic structure. For any commutative ring R , each element is considered a vertex in the zero divisor graph, and two vertices are adjacent precisely when the product of their corresponding elements equals zero [2,4]. Since its introduction, zero divisor graphs have been widely studied to visualize relationships among ring elements and to analyze algebraic structures [1,3,19,20].

In recent years, considerable attention has been devoted to the investigation of topological indices on zero divisor graphs of various commutative rings. Several studies have focused on computing Zagreb indices, Randić-type indices, and related invariants for zero divisor graphs



arising from rings of integers modulo prime powers and their generalizations [7–12]. Computational approaches, including implementations using Python, have also been developed to support the calculation and verification of these indices [14].

Furthermore, Maulana *et al.* have investigated closed formulas for several topological indices of zero divisor graphs associated with the ring of integers modulo prime powers and their direct products, and examined how these indices relate to fundamental graph parameters such as the number of vertices and edges [15]. Related works have also explored topological indices on graphs associated with algebraic structures beyond zero divisor graphs, including unit graphs, power graphs, and non-coprime graphs, highlighting the growing interaction between algebraic graph theory and topological indices [16–18].

However, most existing studies are restricted to the zero divisor graph itself. The study of topological indices on the line graph associated with a zero divisor graph remains relatively unexplored. Motivated by this gap, the present research extends the analysis to the line graph of the zero divisor graph of the ring of integers modulo prime powers, aiming to determine explicit formulas for selected topological indices and to investigate the relationship between the indices of the original zero divisor graph and those of its line graph.

Definition 1.1.[4] *Let R be a commutative ring. The zero divisor graph of R , denoted by ΓR , is a simple graph of the vertex set R and two distinct vertices x and y are joined by an edge whenever $xy = 0$.*

Definition 1.2. [24] *Let G be a connected graph. The Hyper-Wiener index of G denoted by $WW(G)$, is defined as*

$$WW(G) = \frac{1}{2} \left(\sum_{u,v \in V(G)} d(u,v) + \sum_{u,v \in V(G)} d(u,v)^2 \right)$$

where $d(u,v)$ is the distance between the unordered pair of vertices u and v .

Definition 1.3. [24] *Let G be a connected graph. The Harary index of G denoted by $H(G)$, is defined as*

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d(u,v)}$$

where $d(u,v)$ is the distance between the unordered pair of vertices u and v .

Definition 1.4. [8] *Let G be a connected graph. Then, the first Zagreb index of G is the sum of squares of the degrees of the vertices of G , written as,*

$$M_1(G) = \sum_{u \in V(G)} \deg(u)^2$$

where $\deg(u)$ is the number of edges connected to vertex u .

For graphs with diameter at most two, the hyper-Wiener and Harary indices admits a simplified closed form depending only on the number of vertices and edges. This observation enables an efficient computation of the hyper-Wiener and Harary index for line graphs of zero divisor graphs considered in this study.

Lemma 1.1. [15] Let G be a simple connected graph with $\text{diam}(G) \leq 2$, then the hyper-Wiener index of G is $\frac{3}{2}|V(G)|(|V(G)| - 1) - 2|E(G)|$.

Lemma 1.2. [15] Let G be a simple connected graph with $\text{diam}(G) \leq 2$, then the Harary index of G is $\frac{1}{4}|V(G)|(|V(G)| - 1) + \frac{1}{2}|E(G)|$.

Throughout this paper, the ring of integers modulo a prime power is denoted by \mathbb{Z}_p^n , where p is a prime number and n is a positive integer. To support the computation of the topological indices in the next section, the following lemma provides a structural property concerning the number of edges of the zero divisor graph.

Lemma 1.3. [15] The number of edges of $\Gamma\mathbb{Z}_p^n$ is $\frac{1}{2} \left((n+1)p^n - np^{n-1} - p^{\lfloor \frac{n-1}{2} \rfloor} \right)$

Furthermore, the next lemma describes the first Zagreb index of the zero divisor graph of \mathbb{Z}_p^n , which will be used in deriving the main results.

Lemma 1.4. [15] Let $\Gamma\mathbb{Z}_p^n$ be the zero divisor graph of \mathbb{Z}_p^n with prime number p and natural number n . Then, the first Zagreb index of $\Gamma\mathbb{Z}_p^n$ is

$$M_1(\Gamma\mathbb{Z}_p^n) = (p^n - 1)^2 + (p^n - p^{n-1}) \left(\frac{p^n - 1}{p-1} - 2 \left\lfloor \frac{n-1}{2} \right\rfloor \right) + p^{\lfloor \frac{n-1}{2} \rfloor} - 1.$$

2. MAIN RESULTS

By applying the previous lemmas, we now obtain the following theorem regarding the hyper-Wiener index of the line graph $L(G)$.

Theorem 2.1. Let G be a graph. If the line graph $L(G)$ is connected and has diameter at most 2, then $WW(L(G)) = \frac{1}{2}|E(G)|(3|E(G)| + 1) - M_1(G)$.

Proof. Two edges e and f in a graph G are adjacent in the line graph $L(G)$ if they share a common vertex in G . Each vertex v in G has $\text{deg}(v)$ incident edges, and from these edges we can form $C(\text{deg}(v), 2)$ distinct pairs of edges that will become edges in $L(G)$.

Therefore, the number of edges in $L(G)$ is:

$$\begin{aligned} |E(L(G))| &= \sum_{v \in V(G)} C(\text{deg}(v), 2) \\ &= \frac{1}{2} \sum_{v \in V(G)} (\text{deg}(v)^2 - \text{deg}(v)) \\ &= \frac{1}{2} M_1(G) - |E(G)| \end{aligned}$$

By using Lemma 1.1, we have

$$\begin{aligned} WW(L(G)) &= \frac{3}{2} |V(L(G))| (|V(L(G))| - 1) - 2|E(L(G))| \\ &= \frac{3}{2} |E(G)| (|E(G)| - 1) - M_1(G) + 2|E(G)| \\ &= \frac{1}{2} |E(G)| (3|E(G)| + 1) - M_1(G). \blacksquare \end{aligned}$$

Similarly, by using Lemma 1.2, the Harary index of the line graph can be formulated as stated in the following theorem.

Theorem 2.2. Let G be a graph. If the line graph $L(G)$ is connected and has diameter at most 2, then $H(L(G)) = \frac{1}{4}|E(G)|(|E(G)| - 3) + \frac{1}{4}M_1(G)$.

Proof. Two edges e and f in a graph G are adjacent in the line graph $L(G)$ if they share a common vertex in G . Each vertex v in G has $\deg(v)$ incident edges, and from these edges we can form $C(\deg(v), 2)$ distinct pairs of edges that will become edges in $L(G)$.

Therefore, the number of edges in $L(G)$ is:

$$\begin{aligned} |E(L(G))| &= \sum_{v \in V(G)} C(\deg(v), 2) \\ &= \frac{1}{2} \sum_{v \in V(G)} (\deg(v)^2 - \deg(v)) \\ &= \frac{1}{2}M_1(G) - |E(G)| \end{aligned}$$

By using Lemma 1.2, we have

$$\begin{aligned} H(L(G)) &= \frac{1}{4}|V(L(G))|(|V(L(G))| - 1) + \frac{1}{2}|E(L(G))| \\ &= \frac{1}{4}|E(G)|(|E(G)| - 1) + \frac{1}{4}M_1(G) - \frac{1}{2}|E(G)| \\ &= \frac{1}{4}|E(G)|(|E(G)| - 3) + \frac{1}{4}M_1(G). \blacksquare \end{aligned}$$

Since the line graph of zero divisor graph of \mathbb{Z}_p^n has diameter at most 2 [15], the following corollaries follow directly from Theorem 2.1 and Theorem 2.2.

Corollary 2.1. The hyper-Wiener index of $L(\Gamma\mathbb{Z}_p^n)$ is

$$\begin{aligned} &\frac{1}{4} \left((n+1)p^n - np^{n-1} - p^{\lfloor \frac{n-1}{2} \rfloor} \right) \left(\frac{3}{2} \left((n+1)p^n - np^{n-1} - p^{\lfloor \frac{n-1}{2} \rfloor} \right) + 1 \right) \\ &\quad - \left((p^n - 1)^2 + (p^n - p^{n-1}) \left(\frac{p^n - 1}{p - 1} - 2 \lfloor \frac{n-1}{2} \rfloor \right) + p^{\lfloor \frac{n-1}{2} \rfloor} - 1 \right). \end{aligned}$$

Corollary 2.2. The Harary index of $L(\Gamma\mathbb{Z}_p^n)$ is

$$\begin{aligned} &\frac{1}{8} \left((n+1)p^n - np^{n-1} - p^{\lfloor \frac{n-1}{2} \rfloor} \right) \left(\frac{1}{2} \left((n+1)p^n - np^{n-1} - p^{\lfloor \frac{n-1}{2} \rfloor} \right) - 3 \right) \\ &\quad + \frac{1}{4} \left((p^n - 1)^2 + (p^n - p^{n-1}) \left(\frac{p^n - 1}{p - 1} - 2 \lfloor \frac{n-1}{2} \rfloor \right) + p^{\lfloor \frac{n-1}{2} \rfloor} - 1 \right). \end{aligned}$$

These results explicitly demonstrate that the Hyper-Wiener and Harary indices of the line graph $L(\Gamma\mathbb{Z}_p^n)$ can be expressed in terms of the number of edges and the first Zagreb index of the original zero divisor graph, thereby highlighting the structural dependence between the original graph and its line graph.

.CONFLICT OF INTEREST

There is no conflict of interest

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