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Exact Sequences in (R,S)-Modules

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Abstract

The concept of modules has been generalized to the structure of (R,S)-bimodules, which has further been extended to (R,S)-modules. Various properties of classical module theory have been developed within the framework of (R,S)-modules. One of the important topics in module theory that can be extended to this setting is the concept of exact sequences. In this paper, we develop the notion of exact sequences in modules to exact sequences in (R,S)-modules. We begin by introducing the definitions of exact sequences and short exact sequences in (R,S)-modules and examine their basic properties. Several examples are provided to illustrate the concepts, highlighting their similarities with classical module theory, as well as crucial differences when operating over non-unital ring structures. In addition, we establish criteria for determining when a short exact sequence of (R,S)-modules splits and present related structural results using commutative diagrams. Furthermore, we present the necessary and sufficient conditions for an (R,S)-module to be Noetherian in a short exact sequence. These results contribute to a deeper understanding of the homological behavior of (R,S)-modules and provide a foundation for further research in this area.
Keywords: homomorphism, exact sequence, Noetherian, (R,S)-module

1. Introduction

Module theory is one of the fundamental areas in abstract algebra that plays an important role in the study of algebraic structures and their relationships. Many important concepts in algebra, particularly in homological algebra, are formulated using modules and their homomorphisms [1]. One of the central notions in this theory is the concept of exact sequences [2,3], which provide a powerful tool for describing the structure of modules and the relationships between them. An exact sequence captures how the image of one homomorphism relates to the kernel of another, allowing mathematicians to study algebraic structures in a systematic and structural way. In particular, short exact sequences are widely used to analyze extensions of modules, decomposition of structures, and various homological properties. These concepts have been extensively studied in the context of classical module theory [4,5].

Along with the development of algebra, module structures have been generalized into more complex forms, such as (R,S)-bimodules and further into (R,S)-modules [6]. This generalization provides a broader framework that allows the interaction of two ring structures within a single module-like object. Consequently, many concepts in classical module theory can be extended and studied in this more general setting. In particular, the concepts of homomorphisms up to the fundamental isomorphism theorems in (R,S)-modules have been studied by Yuwaningsih et al. [7]. Moreover, the concepts of Noetherian and Artinian (R,S)-modules have also been discussed in [8]. Despite these developments, the study of exact sequences in (R,S)-modules has not been as extensively developed as in the classical case.

Therefore, it becomes important to define and investigate exact sequences in the context of (R,S)-modules, including their fundamental properties and structural characteristics. In this paper, we develop the notion of exact sequences and short exact sequences in (R,S)-modules and examine their basic properties. We also provide several examples to illustrate the concepts, highlighting their similarities with classical module theory, as well as crucial differences when operating over non-unital ring structures. Furthermore, we establish criteria for determining when a short exact sequence of (R,S)-modules splits and present related structural results using commutative diagrams. In addition, we derive necessary and sufficient conditions for an (R,S)-module to be Noetherian in the context of short exact sequences.

Throughout this paper, R and S are arbitrary rings and M is an abelian group under addition, unless stated otherwise.

2. The Definition of Exact Sequences in (R,S)-Module

In this section, we present the definition of exact sequences in (R,S)-modules along with short exact sequences and their several examples.

Definition 2.1. Let R and S be rings, $\{M_i\}$ a family of (R,S)-module, and $f_i: M_{i-1} \to M_i$ an (R,S)-module homomorphisms for each i. A sequences

$$\dots \to M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \to \cdots$$

is said to be exact at M_i if $Im(f_i) = Ker(f_{i+1})$. Moreover, the sequence is said to be exact if it is exact at each M_i.

We now present the definition of a short exact sequence in (R,S)-modules.

Definition 2.2. Let R, S be rings, M, M_1, M_2 (R,S)-modules, and $f: M_1 \to M$, $g: M \to M_2$ (R,S)-module homomorphisms. A sequence

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$$

is said to be short exact if it is exact at M, M_1, and M_2.

Example 2.3. Let $M_{2\times1}(\mathbb{Q})$ be an $(M_{2\times2}(\mathbb{Q}),\mathbb{Q})$-module and $N = \left\{\begin{bmatrix}0\\y\end{bmatrix} \mid y \in \mathbb{Q}\right\}$ an $(M_{2\times2}(\mathbb{Q}),\mathbb{Q})$-submodule of $M_{2\times1}(\mathbb{Q})$. The sequence

$$0 \to N \xrightarrow{i} M_{2\times1}(\mathbb{Q}) \xrightarrow{\pi} \mathbb{Q} \to 0$$

is a short exact sequence with i is the inclusion map and π is the projection map. □

Example 2.4. Let R, S be rings and M_1, M_2 (R,S)-modules. We define (R,S)-module homomorphisms $f: M_1 \to M_1 \times M_2$ by $f(x) = (x, 0_{M_2})$ and $g: M_1 \times M_2 \to M_2$ by $g((x,y)) = y$. We can show that a sequence

$$0 \to M_1 \xrightarrow{f} M_1 \times M_2 \xrightarrow{g} M_2 \to 0$$

is a short exact sequence. □

3. Some Properties of Exact Sequences in (R,S)-Module

In this section, we present several properties of exact sequences and short exact sequences in (R,S)-modules. According to the definition of a short exact sequence, we can derive the following property.

Theorem 3.1. Let R, S be rings, M, M_1, M_2 (R,S)-modules, and $f: M_1 \to M$, $g: M \to M_2$ (R,S)-module homomorphisms.

a) The sequence $0 \to M_1 \xrightarrow{f} M$ is exact at M_1 if and only if f is injective.

b) The sequence $M \xrightarrow{g} M_2 \to 0$ is exact at M_2 if and only if g is surjective.

Proof. Let R, S be rings, M, M_1, M_2 (R,S)-modules, and $f: M_1 \to M$, $g: M \to M_2$ (R,S)-module homomorphisms.

a) Let the sequence $0 \xrightarrow{\theta} M_1 \xrightarrow{f} M$ is exact at M_1. Clearly that $\{0_{M_1}\} = Im(\theta) = Ker(f)$. Thus, we obtain f is a injective function. Conversely, let f is an injective function. We have $Ker(f) = \{0_{M_1}\}$. On the other hand, $Im(\theta) = \{0_{M_1}\}$. Therefore, it is proven that $Im(\theta) = Ker(f)$. Hence, the sequence $0 \to M_1 \xrightarrow{f} M$ is exact at M_1.

b) Let the sequence $M \xrightarrow{g} M_2 \xrightarrow{\theta} 0$ is exact at M_2. Clearly that $Im(g) = Ker(\theta)$. Since θ is a zero homomorphism, we obtain $Ker(\theta) = M_2$. Thus, g is a surjective function. Conversely, let g is a surjective function. So, we obtain $Im(g) = M_2$. Since $Ker(\theta) = M_2$, we have $Im(g) = Ker(\theta)$. Thus, the sequence $M \xrightarrow{g} M_2 \to 0$ is exact at M_2. □

From Theorem 3.1 above, we can derive a theorem that describes a characterization of short exact sequences.

Theorem 3.2. Let R, S be rings, M, M_1, M_2 (R,S)-modules, and $f: M_1 \to M$, $g: M \to M_2$ (R,S)-module homomorphisms. The sequence

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$$

is a short exact sequence if and only if f is injective, g is surjective, and $Im(f) = Ker(g)$.

Proof. Let $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ is a short exact sequence. Thus, this sequence exact at M_1, M, and M_2. Since it is exact at M_1, based on Theorem 3.1 we obtain f is injective. Since it is exact at M_2, based on Theorem 3.1 we get g is surjective. Furthermore, since it is exat at M, we obtain $Im(f) = Ker(g)$. Conversely, it is obvious by using Theorem 3.1. and Definition 2.1. □

Note that, by using the fundamental isomorphism theorem [7], if the sequence

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$$

is exact, then we obtain

$$M/_{Ker(g)} \cong M_2.$$

Since $Im(f) = Ker(g)$, we get $M/_{Im(f)} \cong M_2$.

Example 3.3. Let $\mathbb{Z}$ be a ring and $\mathbb{Z}/_{3\mathbb{Z}}, \mathbb{Z}/_{6\mathbb{Z}}, \mathbb{Z}/_{2\mathbb{Z}}$ be $(\mathbb{Z}, \mathbb{Z})$-modules. The sequence

$$0 \to \mathbb{Z}/_{3\mathbb{Z}} \xrightarrow{f} \mathbb{Z}/_{6\mathbb{Z}} \xrightarrow{g} \mathbb{Z}/_{2\mathbb{Z}} \to 0$$

is a short exact sequence by definition $f(a + 3\mathbb{Z}) = 2a + 3\mathbb{Z}$ and $g(b + 6\mathbb{Z}) = (b \bmod 2) + 2\mathbb{Z}$ for all $a + 3\mathbb{Z} \in \mathbb{Z}/_{3\mathbb{Z}}$ and $b + 6\mathbb{Z} \in \mathbb{Z}/_{6\mathbb{Z}}$. By using the fundamental isomorphism theorem [7], we obtain $\mathbb{Z}/_{2\mathbb{Z}} \cong {(\mathbb{Z}/_{6\mathbb{Z}})}\Big/_{(2\mathbb{Z}/_{6\mathbb{Z}})}$. □

Next, we present several properties of short exact sequences.

Proposition 3.4. Let R, S be rings and M_1, M_2 (R, S)-modules. If $f: M_1 \to M_2$ is an (R, S)-module homomorphism, then the sequence $0 \xrightarrow{\varphi} ker(f) \to M_1 \to Im(f) \xrightarrow{\phi} 0$ is a short exact sequence.

Proof. Define the map $i: \ker(f) \to M_1$ by $i(m) = m$ for every $m \in \ker(f)$ and $\pi: M_1 \to Im(f)$ by $\pi(m_1) = f(m_1)$ for every $m_1 \in M_1$.

a) Let any $m, m' \in \ker(f)$, $r \in R$, and $s \in S$. Observe that$i(m + m') = m + m' = i(m) + i(m')$ and $i(rms) = rms = ri(m)s$. Thus i is an (R, S)-module homomorphism. Next, let $x \in Ker(i)$. Thus, $i(x) = 0_{M_1}$ so that we have $x = 0_{M_1}$. Hence, $Ker(i) = \{0_{M_1}\}$. Therefore, we obtain that i is an (R, S)-module monomorphism.
b) Let any $m_1, m_1' \in M_1$, $r \in R$, and $s \in S$. Observe that $\pi(m_1 + m_1') = f(m_1 + m_1') = f(m_1) + f(m_1')$ and $\pi(rms) = f(rms) = rf(m)s$. Thus θ is an (R, S)-module homomorphism. Next, let any $m_2 \in Im(f)$. Consequently, there exists $m_1 \in M_1$ such that satisfy $f(m_1) = \pi(m_1) = m_2$. Thus, we obtain $Im(\pi) = Im(f)$. Therefore, we obtain that θ is an (R, S)-module epimorphism.
c) Let any $m_1 \in Im(i)$. There exists $x \in Ker(f)$ such that $i(x) = m_1$, so that we obtain $x = m_1$. Since $x \in Ker(f)$, we obtain $\pi(m_1) = f(m_1) = f(x) = 0_{M_1}$. Thus, we obtain $Im(i) \subseteq Ker(\pi)$. In another side, let any $x \in \ker(\pi)$. Thus, we have $\pi(x) = 0_{M_2}$, so that $\pi(x) = f(x) = 0_{M_2}$. Consequently, we obtain $x \in Ker(f)$. Based on the definition of inclusion map i, we obtain $i(x) = x$. Hence, $Ker(\theta) \subseteq Im(i)$. Thus, it is proven that $Im(i) = Ker(\theta)$ so that this sequence exact at M_1.

Hence, the sequence $0 \xrightarrow{\varphi} ker(f) \to M_1 \to Im(f) \xrightarrow{\phi} 0$ is a shoret exact sequence. □

Proposition 3.5. Let R, S be rings and M_1, M_2, M_3, M, N (R, S)-modules. If $0 \to M_1 \to M \xrightarrow{f} N \to 0$ and $0 \to N \xrightarrow{g} M_2 \to M_3 \to 0$ are short exact sequences, then $0 \to M_1 \to M \xrightarrow{g \circ f} M_2 \to M_3 \to 0$ is an exact sequence.

Proof. Let a sequence $0 \to M_1 \xrightarrow{i} M \xrightarrow{g \circ f} M_2 \xrightarrow{p} M_3 \to 0$. Clearly that this sequence exact at M_1 and M_3. We will show that this sequence exact at M and M_2.

a) Let any $x \in Im(i)$. There exists $m_1 \in M_1$ such that $i(m_1) = x$. Since $0 \to M_1 \to M \xrightarrow{f} N \to 0$ is a short exat sequence, we get $f\big(i(m_1)\big) = f(x) = 0_N$. Consequently, $(g \circ f)(x) = g\big(f(x)\big) = g(0_N) = 0_{M_2}$. Thus, $x \in Ker(g \circ f)$ so that we get $Im(i) \subseteq Ker(g \circ f)$. In another side, let $x \in Ker(g \circ f)$. We obtain $(g \circ f)(x) = g\big(f(x)\big) = 0_{M_2}$. Since g is injective, we get $f(x) = 0_N$. So, $x \in Ker(f)$. Since $Ker(f) = Im(i)$, we obtain $x \in Im(i)$. Hence, we get $Ker(g \circ f) \subseteq Im(i)$ so that $Im(i) = Ker(g \circ f)$. Thus, it is proven that $0 \to M_1 \to M \xrightarrow{g\circ f} M_2 \to M_3 \to 0$ is exact at M.

b) Let any $y \in Im(g \circ f)$. There exists $m \in M$ such that $(g \circ f)(m) = g\big(f(m)\big) = y$. Since $0 \to N \xrightarrow{g} M_2 \to M_3 \to 0$ is exact at M_2, we obtain $p\big(g(n)\big) = 0_{M_3}$ for each $n \in N$. Consequently, we obtain $p(y) = p\Big(g\big(f(m)\big)\Big) = 0_{M_3}$. Thus, we obtain $y \in Ker(p)$ or $Im(g \circ f) \subseteq Ker(p)$. In another side, let $y \in Ker(p)$. Since $0 \to N \xrightarrow{g} M_2 \to M_3 \to 0$ is exact at M_2, we get $Ker(p) = Im(g)$. Thus, there exists $n \in N$ such that $g(n) = y$. Since $0 \to M_1 \to M \xrightarrow{f} N \to 0$ is exact, we obtain f is surjective. Thus, there exists $m \in M$ such that $f(m) = n$. Moreover, we obtain $y = g\big(f(m)\big) = (g \circ f)(m)$. Hence, $y \in Im(g \circ f)$ so that $Ker(p) \subseteq Im(g \circ f)$. Thus, we obtain $Im(g \circ f) = Ker(p)$. Therefore, we get $0 \to M_1 \to M \xrightarrow{g\circ f} M_2 \to M_3 \to 0$ is exact at M_2.

Hence, it is proven that $0 \to M_1 \to M \xrightarrow{g\circ f} M_2 \to M_3 \to 0$ is an exact sequence. □

In the following, we present a property of exact sequences in (R, S)-modules that forms a commutative diagram.

Proposition 3.6. Consider the following commutative diagram of exact sequences of (R, S)-modules, together with (R, S)-module homomorphisms α, β, and γ as follow:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N & \longrightarrow & N_2 & \longrightarrow & 0 \end{array}$$

a) If α and γ are monomorphsms, then β is monomorphism.
b) If α and γ are epimorphsms, then β is epimorphism.
c) If α and γ are isomorphsms, then β is isomorphism.

Proof. Let $f: M_1 \to M$, $g: M \to M_2, f': N_1 \to N$, and $g': N \to N_2$ are (R, S)-module homomorphisms.

a) Let any $m \in Ker(\beta)$. We have $\beta(m) = 0_N$. Since that diagram is commutative, we obtain $\gamma\big(g(m)\big) = g'\big(\beta(m)\big)$, so that

$$\gamma\big(g(m)\big) = g'\big(\beta(m)\big) = g'(0_N) = 0_{N_2}.$$

On the other hand, since γ is a monomorphism, we obtain $g(m) = 0_{M_2}$. This means that $m \in Ker(g)$. Since $Ker(g) = Im(f)$, we obtain $m = f(m_1)$ for some $m_1 \in M_1$. Furthermore, since that diagram is commutative, we obtain

$$f'\big(\alpha(m_1)\big) = \beta\big(f(m_1)\big) = \beta(m) = 0_N.$$

Since f' and α are monomorphism, $\alpha(m_1) = 0_{N_1}$ so that $m_1 = 0_{M_1}$. Hence, we obtain $m = f(m_1) = 0_M$. Thus, $Ker(\beta) = \{0_M\}$. Hence, it is proved that β is monomorphism.

b) Let any $n \in N$. Then, $n_2 = g'(n) \in N_2$. Since γ is epimorphism and $n_2 \in N_2$, there exists $m_2 \in M_2$ such that $\gamma(m_2) = n_2$. In the other hand, since g is epimorphism, there exists $m' \in M$ such that $g(m') = m_2$. Consequently, we obtain

$$g'(\beta(m')) = \gamma(g(m')) = \gamma(m_2) = n_2.$$

Since $g'(n) = n_2$ and $g'(\beta(m')) = n_2$, we get

$$g'(n - \beta(m')) = g'(n) - g'(\beta(m')) = n_2 - n_2 = 0_{N_2}.$$

So, $n - \beta(m') \in Ker(g')$. Since $Ker(g') = Im(f')$, there exists $n_1 \in N_1$ such that $f'(n_1) = n - \beta(m')$. Furthermore, since α is epimorphism, there exists $m_1 \in M_1$ such that $\alpha(m_1) = n_1$, for some $n_1 \in N_1$. Thus, we obtain

$$f'(n_1) = f'(\alpha(m_1)) = n - \beta(m').$$

Based on the commutativity of the diagram, we obtain

$$f'(\alpha(m_1)) = \beta(f(m_1)) = n - \beta(m') \Rightarrow n = \beta(m' + f(m_1)).$$

Thus, for each $n \in N$ there exists $m = m' + f(m_1) \in M$ such that $\beta(m) = n$. Hence, it is proven that β is epimorphism.

c) Since β is monomorphism (from a) and β is epimorphims (from b), then we obtain β is isomorphism. □

Next, we present the definition of a split exact sequence along with its examples.

Definition 3.7. Let R, S be rings, M, M_1, M_2 (R,S)-modules, and $f: M_1 \to M$, $g: M \to M_2$ (R,S)-module homomorphisms. The short exact sequence

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$$

is said to be split if $Im(f) = Ker(g)$ is a direct summand of M.

Example 3.8. Based on Example 3.3. we know that

$$Im(f) = Ker(g) = {}^{2\mathbb{Z}}/_{6\mathbb{Z}} = \{0 + 6\mathbb{Z}, 2 + 6\mathbb{Z}, 4 + 6\mathbb{Z}\}.$$

It is obvious that ${}^{2\mathbb{Z}}/_{6\mathbb{Z}}$ is a direct summand of ${}^{\mathbb{Z}}/_{6\mathbb{Z}}$. □

Example 3.9. Let $R = 2\mathbb{Z}$ and $S = 2\mathbb{Z}$ be non-unital rings. Consider $M = 2\mathbb{Z}$ as an (R,S)-module with standard multiplication. Let $M_1 = 4\mathbb{Z}$ be an (R,S)-submodule of M, and $M_2 = {}^{M}/_{M_1} = {}^{2\mathbb{Z}}/_{4\mathbb{Z}}$ be a factor module. We can construct the following short exact sequence:

$$0 \to 4\mathbb{Z} \xrightarrow{i} 2\mathbb{Z} \xrightarrow{\pi} {}^{2\mathbb{Z}}/_{4\mathbb{Z}} \to 0$$

where i is the inclusion map and π is the natural projection. Since $Im(i) = Ker(\pi) = 4\mathbb{Z}$, then we obtain that this short exact sequence is split. □

Although many properties naturally extend to (R,S)-modules over unital rings, significant structural differences emerge when the underlying rings lack a multiplicative identity. Unlike classical theory where splitting exact sequences yield rigid direct sum decompositions, non-unital ring actions can collapse these dualities. The following example explicitly highlights these homological differences by utilizing a non-unital ring structure.

Example 3.10. Let $R = 2\mathbb{Z}$ and $S = \mathbb{Z}$. Consider $M = \mathbb{Z}_4 \times \mathbb{Z}_2$ be an (R, S)-module with standar multiplication. Let $M_1 = \{(\bar{0}, \bar{0}), (\bar{2}, \bar{0})\} \cong \mathbb{Z}_2$ and $M_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Next, consider the following short exact sequence of (R, S)-modules:

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$$

where f is the inclusion map $f(\bar{a}) = (\bar{a}, \bar{0})$ and g is the canonical projection map $g(\bar{x}, \bar{y}) = (\bar{x}\ mod\ 2,\ \bar{y})$. Unlike classical theory over unital rings where splitting exact sequences yield rigid direct sum decompositions, this non-unital setting exhibits two key structural differences:

a) The left R-action completely annihilates the module (i.e., $R \cdot M = \bar{0}$), preventing the use of standard ring-coefficient tools such as bases or projectivity from the left side.
b) Although the sequence splits as an abelian group and a right unital $\mathbb{Z}$-module, the absence of a multiplicative identity in R implies that M cannot decompose into a direct sum of *unital* (R, S)-modules, as the classical condition $1_R \cdot m = m$ fails. □

The following is an important theorem that serves as a criterion for determining whether a short exact sequence is split.

Theorem 3.11. Let R, S be rings, M, M_1, M_2 (R, S)-modules, and $f: M_1 \to M$, $g: M \to M_2$ (R, S)-module homomorphisms. If the sequence

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$$

is a short exact sequence, then the following three statements are equivalent:

a) There exists an (R, S)-module homomorphism $\alpha: M \to M_1$ such that $\alpha \circ f = id_{M_1}$, where id_{M_1} is an identity homomorphism from M_1 to M_1.
b) There exists an (R, S)-module homomorphism $\beta: M_2 \to M$ such that $g \circ \beta = id_{M_2}$, where id_{M_2} is an identity homomorphism from M_2 to M_2.
c) The sequence $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ is a split exact sequence and satisfy

$$\begin{aligned} M &\cong Im(f) \oplus Ker(\alpha) \\ &\cong Ker(g) \oplus Im(\beta) \\ &\cong M_1 \oplus M_2 \end{aligned}$$

Proof. $a) \Rightarrow b)$. Let $\alpha: M \to M_1$ is an (R, S)-module homomorphism such that $\alpha \circ f = \mathrm{id}_{M_1}$. Let any $y \in M_2$. Since g is epimorphism, there exists $m \in M$ such that $g(m) = y$. Next, we define $\beta: M_2 \to M$ by $\beta(y) = m - f\big(\alpha(m)\big)$. We can show that β is an (R, S)-module homomorphism. Moreover, let $x \in M_2$ and $m \in M$ with $g(m) = x$. Consoder that

$$\begin{aligned} (g \circ \beta)(x) &= g\big(\beta(x)\big) \\ &= g\big(m - f\big(\alpha(m)\big)\big) \\ &= g(m) - g\Big(f\big(\alpha(m)\big)\Big) \\ &= x - 0_{M_2} \\ &= x \end{aligned}$$

$$= id_{M_2}(x).$$

Thus, there exists an (R,S)-module homomorphism $\beta: M_2 \to M$ such that $g \circ \beta = id_{M_2}$.

$b) \Rightarrow c)$. Let any $m \in M$. Then, m can be written by

$$m = \big(m - \beta(g(m))\big) + \beta(g(m)).$$

Consider that

$$g\left(m - \beta(g(m))\right) = g(m) - g\left(\beta(g(m))\right) = g(m) - \mathrm{id}_{M_2}(g(m)) = 0_{M_2}.$$

Thus, $m - \beta(g(m)) \in Ker(g)$ and $\beta(g(m)) \in Im(\beta)$. So, we obtain $M = Ker(g) + Im(\beta)$. Next, let $x \in Ker(g) \cap Im(\beta)$. We obtain $x \in Ker(g)$ and $x \in Im(\beta)$, so that we get $g(x) = 0_{M_2}$ and $\beta(y) = x$, for some $y \in M_2$. Thus, $0_{M_2} = g(x) = g(\beta(y)) = y$. Since $y = 0_{M_2}$, we have $x = \beta(0_{M_2}) = 0_M$. Thus, $Ker(g) \cap Im(\beta) = \{0_M\}$. Hence, we obtain $M = Ker(g) \oplus Im(\beta)$. Moreover, the sequence $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ is exact, we have $Im(f) = Ker(g)$ and f is monomorphism. Consequently, we obtain $Im(f) \cong M_1$. Since $g \circ \beta = \mathrm{id}_{M_2}$, we obtain $Im(\beta) \cong M_2$. Thus, it is proven that $M \cong M_1 \oplus M_2$.

$c) \Rightarrow a)$. Let the sequence $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ is a split exact sequence and satisfy $M \cong M' \oplus M"$, where $M' = Ker(g)$ and $M" = Im(f)$. Let projection maps $\pi_1: M \to M'$ and $\pi_2: M \to M"$ and an inclusion map $i: M" \to M$. We obtain $\pi_1 \circ f: M_1 \to M'$ and $g \circ i: M" \to M_2$ are (R,S)-module isomorphism. Moreover, we define the map $\alpha: M \to M_1$ by $\alpha(a,b) = ((\pi_1 \circ f)^{-1} \circ \pi_1)(a,b)$ for all $(a,b) \in M$. We can show that α is an (R,S)-module homomorphism. Let any $(a,b),(c,d) \in M$, $r \in R$, and $s \in S$. We obtain

$$\begin{aligned}
\alpha((a,b)+(c,d)) &= ((\pi_1 \circ f)^{-1} \circ \pi_1)((a,b)+(c,d)) \\
&= ((\pi_1 \circ f)^{-1} \circ \pi_1)(a+c, b+d) \\
&= (\pi_1 \circ f)^{-1}(a+c) \\
&= (\pi_1 \circ f)^{-1}(a) + (\pi_1 \circ f)^{-1}(c) \\
&= ((\pi_1 \circ f)^{-1} \circ \pi_1)(a,b) + ((\pi_1 \circ f)^{-1} \circ \pi_1)(c,d) \\
&= \alpha(a,b) + \alpha(c,d)
\end{aligned}$$

dan

$$\begin{aligned}
\alpha(r(a,b)s) &= ((\pi_1 \circ f)^{-1} \circ \pi_1)(r(a,b)s) \\
&= ((\pi_1 \circ f)^{-1} \circ \pi_1)(ras, rbs) \\
&= (\pi_1 \circ f)^{-1}(ras) \\
&= r(\pi_1 \circ f)^{-1}(a)s \\
&= r((\pi_1 \circ f)^{-1} \circ \pi_1)(a,b)s
\end{aligned}$$

Thus, we get that α is an (R,S)-module homomorphism. Furthermore, since $\pi_1 \circ f$ is isomorhism, we obtain

$$\alpha \circ f = (\pi_1 \circ f)^{-1} \circ \pi_1 \circ f = (\pi_1 \circ f)^{-1} \circ (\pi_1 \circ f) = id_{M_1}.$$

Hence, it is proven that there exists an (R,S)-module homomorphism $\alpha: M \to M_1$ such that $\alpha \circ f = id_{M_1}$. □

Before proceeding to the next property, we need this following lemma.

Lemma 3.12. Let R, S be rings, M, M_1, M_2 (R,S)-modules, and $f: M_1 \to M$, $g: M \to M_2$ (R,S)-module homomorphisms, and the short exact sequence

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0.$$

If for submodule $M' \subseteq M" \subseteq M$ satisfy $f(M_1) \cap M' = f(M_1) \cap M"$ and $g(M') = g(M")$, then we obtain $M' = M"$.

Proof. Let any $m \in M"$. Then, we have $g(m) \in g(M") = g(M')$ so that there exists $n \in M'$ such that $g(m) = g(n)$. Then, $g(m-n) = 0_N$ so that $m - n \in M' \cap Ker(g) = M' \cap f(M_1) = M" \cap f(M_1)$. It follows that $m - n \in M'$, so that $m \in M'$. This show that $M" \subseteq M'$. Thus, we obtain $M' = M"$. □

Next, we present the necessary and sufficient conditions for an (R,S)-module to be Noetherian in a short exact sequence.

Proposition 3.13. Let R, S be rings, M, M_1, M_2 (R,S)-modules, and $f: M_1 \to M$, $g: M \to M_2$ (R,S)-module homomorphisms, and the short exact sequence

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0.$$

Then, M is a Noetherian (R,S)-module if and only if M_1 and M_2 are Noetherian (R,S)-modules.

Proof. Let the ascending chain of (R,S)-submodules $\{M_{1i}\}_{i=1}^{\infty}$ in M_1. Then, we obtain the ascending chain (R,S)-submodules $\{f(M_{1i})\}_{i=1}^{\infty}$ in M. Since M is Noetherian, there exists a positive integer n such that $f(M_{1n}) = f\big(M_{i(n+1)}\big) = \cdots$. Applying f^{-1} to both sides, we obtain

$$f^{-1}(f(M_{1n})) = f^{-1}(f\big(M_{1(n+1)}\big)) = \cdots.$$

Since f is injective, we get $f^{-1}(f(M_{1i})) = M_{1i}$ for each i. Thus, we obtain $M_{1n} = M_{1(n+1)} = \cdots$ so that it is proven that M_1 is a Noetherian (R,S)-module. Similarly, let the ascending chain condition of (R,S)-submodules $\{M_{2i}\}_{i=1}^{\infty}$ in M_2. Then, we obtain the asecnding chain condition $\{g^{-1}(M_{2i})\}_{i=1}^{\infty}$ in M. Since M is Noetherian, there exists a positive integer p such that $g^{-1}\big(M_{2p}\big) = g^{-1}\big(M_{2(p+1)}\big) = \cdots$. Applying g to both sides, we get

$$g(g^{-1}\big(M_{2p}\big)) = g(g^{-1}\big(M_{2(p+1)}\big)) = \cdots.$$

Since g is surjective, we get $g(g^{-1}(M_{1i})) = M_{1i}$ for each i. Hence, we obtain $M_{2p} = M_{2(p+1)} = \cdots$ so that it is proven that M_2 is a Noetherian (R,S)-module. Conversely, let $\{N_i\}_{i=1}^{\infty}$ is an ascending chain of (R,S)-submodules of M. Then, identifying $f(M_1)$ with M_1 (which can be done, since f is injective), and taking intersections, we obtain a chain of the form

$$M_1 \cap N_1 \subseteq M_1 \cap N_2 \subseteq \cdots$$

of (R,S)-submodules in M_1. Similarly, applying g we obtain a chain

$$g(N_1) \subseteq g(N_2) \subseteq \cdots$$

of (R,S)-submodules on M_2. Since M_1 and M_2 are Noetherian, there exists a positive integer n such that

$$M_1 \cap N_n = M_1 \cap N_{n+1} = \cdots$$

and

$$g(N_n) = g(N_{n+1}) = \cdots$$

By using Lemma 3.12, we obtain $N_n = N_{n+1} = \cdots$, so that it is proven that M is a Noetherian (R, S)-module. □

In classical module theory, the splitting of a short exact sequence is heavily dependent on the existence of a single ring identity. However, the simultaneous actions of two distinct rings in (R, S)-modules allow for a unique homological interaction when one of the ring actions behaves trivially. The following theorem establishes a specialized splitting criterion that links this property directly to the annihilator of the ring action.

Theorem 3.14. Let R and S be rings with identities. Let

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$$

be a short exact sequence of (R, S)-modules. Suppose that the right S-action on M_2 is trivial (i.e., $m_2 \cdot s = 0$ for all $m_2 \in M_2$, $s \in S$). Then the sequence splits as (R, S)-modules if and only if it splits as left R-modules and there exists an R-module homomorphism $\beta: M_2 \to M$ such that $g \circ \beta = id_{M_2}$ and $Im(\beta) \subseteq Ann_M(S)$, where $Ann_M(S) = \{m \in M \mid m \cdot s = 0, \forall s \in S\}$.

Proof. Suppose the short exact sequence splits as (R, S)-modules. By Theorem 3.11, there exists an (R, S)-module homomorphism $\beta: M_2 \to M$ such that $g \circ \beta = id_{M_2}$. Since S has an identity, we can assume that β is a right S-module homomorphism. So, we have $\beta(m_2 \cdot s) = \beta(m_2) \cdot s$ for all $m_2 \in M_2$ and $s \in S$. Given that the right S-action on M_2 is trivial, it follows that $\beta(0) = \beta(m_2) \cdot s = 0$. Thus, $\beta(m_2) \cdot s = 0$ for all $m_2 \in M_2$, which implies $Im(\beta) \subseteq Ann_M(S)$. Since any (R, S)-homomorphism is inherently a left R-homomorphism, the sequence naturally splits as left R-modules. Conversely, assume the sequence splits as left R-modules via a left R-module homomorphism $\beta: M_2 \to M$ such that $g \circ \beta = id_{M_2}$ and $Im(\beta) \subseteq Ann_M(S)$. To show that the sequence splits as (R, S)-modules, we must verify that β preserves the right S-action. For any $m_2 \in M_2$ and $s \in S$, since the right S-action on M_2 is trivial, we have $\beta(m_2 \cdot s) = \beta(0) = 0$. On the other hand, since $Im(\beta) \subseteq Ann_M(S)$, we have $\beta(m_2) \in Ann_M(S)$, which implies $\beta(m_2) \cdot s = 0$. Therefore, $\beta(m_2 \cdot s) = \beta(m_2) \cdot s$, proving that β is a right S-module homomorphism. Hence, β is a well-defined (R, S)-module homomorphism, and by Theorem 3.11, the sequence splits as (R, S)-modules. □

4. Conclusion

In this paper, we have developed the concept of exact sequences in the setting of (R, S)-modules as an extension of classical module theory. In particular, we introduced the notions of exact sequences and short exact sequences in (R, S)-modules and examined their fundamental properties.
Several examples are provided to illustrate the concepts, highlighting their similarities with classical module theory, as well as crucial differences when operating over non-unital ring structures. Furthermore, we established criteria for determining when a short exact sequence of (R, S)-modules splits and presented related structural results using commutative diagrams. In addition, we obtained necessary and sufficient conditions for an (R, S)-module to be Noetherian in the context of short exact sequences. These results show that the theory of exact sequences can be naturally extended to (R, S)-modules and provide a useful framework for further investigation, particularly in the study of homological properties in this generalized setting.

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