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Homoderivations in Prime Near Rings

Gusti Ayu Dwi Yanti[1*], Sandy Salomo Saruan[2]

[1] Department of Mathematics, Universitas Gadjah Mada
[2] Department of Mathematics, Institut Teknologi Batam
***Email**: [1] gustiayu9d@gmail.com, [2] sandy@iteba.ac.id*
[*]Corresponding Author
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Abstract

Homoderivations represent a generalized differential operator that combines multiplicative and derivational properties, making them significant in studying algebraic structures with hybrid operations. This paper initiates the systematic study of homoderivations on prime near rings, an algebraic structure where only left distributivity holds. We establish the foundational Leibniz formula and investigate when the existence of such mapping forces the underlying near ring to become commutative. Specificall, we prove that if a 2-torsion free prime near ring admits a non zero homoderivation h with $h([x, y]) = 0$ or $h(x \circ y) = 0$, then N is a commutative ring. Furthermore, we characterize the center of the near ring through the behavior of homoderivations on square closed Lie ideals. These results have implications for understanding the interplay between ring theoritic derivation and multiplicative mappings in non ring.

Keywords: homoderivation, prime near ring, commutativity theorem, lie structure

Abstrak

Homoderivasi merupakan operator diferensial tergeneralisasi yang menggabungkan sifat-sifat multiplikatif dan derivasional, sehingga signifikan dalam mempelajari struktur aljaba dengan operasi *hybrid.* Artikel ini memulai kajian sistematis tentang homoderivasi pada near ring prima, yaitu struktur aljabar dengan hanya distributivitas kiri yang berlaku. Diterapkan aturan dasar Leibniz dan mengivestigasi keberadaan pemetaan tersebut yang memaksa near ring menjadi komutatif. Secara khusus, kami membuktikan bahwa jika near ring prima bebas torsi-2 memiliki homoderivasi tidak nol h dengan $h([x, y]) = 0$ atau $h(x \circ y) = 0$, maka N merupakan ring komutatif. Selanjutnya, kami mengkarakterisasi pusat near ring melalui perilaku homoderivasi pada *square closed Lie ideal*. Hasil-hasil ini memiliki implikasi untuk memahami interaksi antara derivasi teori ring dan pemetaan multiplikatif pada non ring.

Kata kunci: homoderivasi, near ring prima, teorema komutativitas, struktur Lie

1. INTRODUCTION AND PRELIMINARIES

A near ring is an algebraic structure $(N,+,\cdot)$ where $(N,+)$ is a group, $(N,\cdot)$ is a semigroup, and left distributivity $x(y+z)=xy+xz$ holds for all $x,y,z\in N$. Unlike rings, near rings do not require right distributivity or additive commutativity. A near ring N is zero symmetric if $x\cdot 0=0$ for all $x\in N$, and prime if $aNb=\{0\}$ implies $a=0$ or $b=0$.

An additive mapping $d: N\to N$ on near ring N is called a derivation if it satisfies the Leibniz rule $d(xy)=d(x)y+xd(y)$ for all $x,y\in N$. [12] proved that a prime ring with a non zero derivation mapping into its center must be commutative. This was extended to near rings by [14] and [4] on derivations in near rings. An additive mapping $d: N\to N$ associated with a pair of automorphisms σ,τ through the identity satisfying $d(xy)=d(x)\tau(y)+\sigma(x)d(y)$ for all $x,y\in N$ is called (σ,τ)-derivation generalizes. [2] in particular obtained the Leibniz formula for (σ,τ)-derivations in near rings and showed that a 2-torsion free prime near ring with a non zero (σ,σ)-derivation annihiliating the Lie product or the Jordan product must be a commutative ring.

[7] introduced an additive mapping $h: R\to R$ on a ring R as a homoderivation if it satisfies $h(xy)=h(x)h(y)+h(x)y+xh(y)$ for all $x,y\in R$. [11] who proved the commutativity of prime rings admitting non zero homoderivations with certain conditions on the Lie product, [1] who investigated homoderivations on left ideals and rings with involution. [8]who studied centrally extended α-homoderivations; and [13] who analyzed generalized homoderivations of prime rings. The pinnacle of recent development was achieved by [9] who comprehensively analyzed the dependence of zero power valued homoderivations satisfying the identity $h_1(x)h_2(y)=h_3(x)h_4(y)$, and proved that square closed Lie ideals satisfying various homoderivational identities must be contained in the center of a prime ring.

[10] proved that homoderivations on semiprime rings are commuting maps on square closed Lie ideals under various differential identity conditions, expanding the understanding of the interaction between homoderivations and Lie structures. [6] explicitly introduced and investigated homoderivations on 3-prime near rings with centralizing constrait, proving commutativity conditions and presenting examples demonstrating the importance of the 3-primeness assumption. [5] analyzed homoderivations and semigroup ideals in the same context.

Based on this background, this research aims to introduce the concept of homoderivation on prime near rings, study its fundamental properties, and investigate conditions that imply the commutativity of near rings and the centrality of Lie ideals. Specifically, the main results sought include the Leibniz formula for homoderivations on near rings, a dependence theorem for zero power valued homoderivations satisfying multiplicative identities, and commutativity conditions obtained from identities on the Lie product and the Jordan product.

2. PRELIMINARIES

Throughout this paper, unless otherwise stated, N will denote a zero symmetric left near ring-with multiplicative center Z. A near ring N is said to be prime if $aNb=\{0\}$ implies that $a=0$ or $b=0$. For any $x,y\in N$, as usual $[x,y]\coloneqq xy-yx$ and $x\circ y\coloneqq xy+yx$ will denote the well known Lie and Jordan products, respectively. An additive subgroup L of N is called a Lie ideal if $[x,r]\in L$ for all $x\in L$ and $r\in N$. If L is a Lie ideal of N and $x^2\in L$ for all $x\in L$, then L is called a square closed Lie ideal.

Definition 2.1 *An additive mapping $d: N\to N$ on a near ring N is called a derivation if it satisfies the Leibniz rule $d(xy)=d(x)y+xd(y)$ for all $x,y\in N$.*

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Definition 2.2 *An additive mapping* $h: N \to N$ *on a near ring* N *is called a homoderivation if it satisfies* $h(xy) = h(x)h(y) + h(x)y + xh(y)$ *for all* $x, y \in N$.

Lemma 2.3 *Let* N *be a prime near ring and* h *be a homoderivation on* N. *If* $h_{|Z} \neq 0$, *then there exists* $0 \neq z \in Z$ *such that* $h(z) \in Z$ *and* $h(z) \neq 0$.

Proof. Since $h_{|Z} \neq 0$, there exists $z_0 \in Z$ such that $h(z_0) \neq 0$. We show that $h(z_0) \in Z$. Let $x \in N$ be arbitrary. Then $h(z_0 x) = h(z_0)h(x) + h(z_0)x + z_0 h(x)$ and $h(x z_0) = h(x)h(z_0) + h(x)z_0 + xh(z_0)$. Since $z_0 \in Z$, we have $z_0 x = x z_0$, and thus $h(z_0 x) = h(x z_0)$. Therefore,

$$h(z_0 h(x) + h(z_0)x + z_0 h(x) = h(x)h(z_0) + h(x)z_0 + xh(z_0).$$

Using the fact that $z_0 \in Z$ and rearranging terms, we obtain

$$h(z_0 x) - xh(z_0) = h(x)h(z_0) - h(z_0)h(x) + h(x)z_0 - z_0 h(x).$$

Since $z_0 \in Z$, the terms $h(x)z_0 - z_0 h(x)$ vanish. Thus, $[h(z_0), x] = [h(x), h(z_0)]$. Replacing x by xy and using the definition of homoderivation, we obtain after simplification that $h(z_0) \in Z$.

Lemma 2.4. Let N be a 2-torsion free prime near ring and d_1, d_2 be derivations on N. If $d_1 d_2 = 0$ then $d_1 = 0$ or $d_2 = 0$.

Proof. Suppose that $d_1 d_2 = 0$, where d_1, d_2 are derivations on the prime near ring N. This means $d_1\big(d_2(x)\big) = 0$ for all $x \in N$. For any $x, y \in N$, since d_2 is a derivation, then $d_2(xy) = d_2(x)y + xd_2(y)$. Apply d_1 to both sides, we get $0 = d_1\big(d_2(x)\big)y + d_2(x)d_1(y) + d_1(x)d_2(y) + xd_1(d_2(y))$. Because $d_1\big(d_2(x)\big) = 0$ and $d_1\big(d_2(y)\big) = 0$, so

$$d_2(x)d_1(y) + d_1(x)d_2(y) = 0 \text{ for all } x, y \in N. \qquad (2.1)$$

Replace y by yz, we have $d_2(x)d_1(yz) + d_1(x)d_2(yz) = 0$. Expand using the Leibniz rule,

$$d_2(x)[d_1(y)z + yd_1(z)] + d_1(x)[d_2(y)z + yd_2(z)] = 0$$

$$d_2(x)d_1(y)z + d_2(x)yd_1(z) + d_1(x)d_2(y)z + d_1(x)yd_2(z) = 0. \qquad (2.2)$$

From Equation (2.2), right multiplied by z, and we obtain

$$d_2(x)d_1(y)z + d_1(x)d_2(y)z = 0. \qquad (2.3)$$

Subtracting Equation (2.3) from Equation (2.2), then

$$d_2(x)yd_1(z) + d_1(x)yd_2(z) = 0 \text{ for all } x, y, z \in N. \qquad (2.4)$$

Assume that $d_1 \neq 0$. Since $d_1 \neq 0$, there exists $a \in N$ such that $d_1(a) \neq 0$. Replace $x = a$, so

$$d_2(a)yd_1(z) + d_1(a)yd_2(z) = 0 \text{ for all } y, z \in N. \qquad (2.5)$$

Replace z by $d_2(w)$ for any $w \in N$, and since $d_1\big(d_2(w)\big) = 0$, we have $d_2(a$ $d_1(a)yd_2\big(d_2(w)\big) = 0$. Therefore $d_1(a)yd_2^2(w) = 0$ for all $y, w \in N$. Since $d_1(a) \neq 0$ and N is prime, then $d_2^2(w) = 0$ for all $w \in N$. That is, $d_2\big(d_2(w)\big) = 0$ for all $w \in N$. From Equation (2.4), replace y by $d_2(u)y$ for any $u \in N$, and for all $y, z, u \in N$ we have

$$d_2(a)d_2(u)yd_1(z) + d_1(a)d_2(u)yd_2(z) = 0$$

$$d_2(a)[-d_1(u)d_2(z)]y + d_1(a)d_2(u)yd_2(z) = 0$$

$$-[-d_1(a)d_2(u)]d_2(z)y + d_1(a)d_2(u)yd_2(z) = 0$$

$$d_1(a)d_2(u)[d_2(z)y + yd_2(z)] = 0$$

$$d_1(a)d_2(u)(d_2(z) \circ y) = 0.$$

Suppose, for the sake of contradiction, that $d_2 \neq 0$. Then there exists $u_0 \in N$ such that $d_2(u_0) \neq 0$. With $u = u_0$, we get $d_1(a)d_2(u_0)(d_2(z) \circ y) = 0$ for all $y, z \in N$. Since $d_1(a) \neq 0$ and $d_2(u_0) \neq 0$ and N is prime, so $d_2(z)y + yd_2(z) = 0$ for all $y, z \in N$. That is, $d_2(z) \circ y = 0$ for all $y, z \in N$. Replace y by $yd_2(z)$, we obtain

$$d_2(z)[yd_2(z)] = -[yd_2(z)]d_2(z)$$

$$d_2(z)yd_2(z) = -yd_2(z)^2$$
$$d_2(z)d_2(z) + d_2(z)d_2(z) = 0$$
$$2d_2(z)^2 = 0\,.$$

If N is 2-torsion free, then $d_2(z)^2 = 0$. Now, from Equation (2.4), replace y by $d_2(z)y$, then

$$d_2(a)d_2(z)yd_1(z) = -d_1(a)d_2(z)yd_2(z)$$
$$d_2(a)d_2(z)yd_1(z) = -d_1(a)[-yd_2(z)]d_2(z)$$
$$d_2(a)d_2(z)yd_1(z) = d_1(a)yd_2(z)^2$$
$$d_2(z)yd_2(z) = -y \cdot 0.$$

So $d_2(z)yd_2(z) = 0$ for all $y, z \in N$. Since N is prime, for each fixed z, from $d_2(z)Nd_2(z) = \{0\}$. We conclude $d_2(z) = 0$. Hence $d_2(z) = 0$ for all $z \in N$, i.e., $d_2 = 0$. This contradicts our assumption that $d_2 \neq 0$. Therefore, if $d_1 \neq 0$, we must have $d_2 = 0$.

Assume that $d_2 \neq 0$. Since $d_2 \neq 0$, there exists $b \in N$ such that $d_2(b) \neq 0$. From equation (2.4) with $x = b$ we have

$$d_2(b)yd_1(z) = -d_1(b)yd_2(z). \tag{2.6}$$

Replace y by $d_2(v)y$ and we obtain

$$d_2(b)d_2(v)yd_1(z) + d_1(b)d_2(v)yd_2(z) = 0 \tag{2.7}$$

From equation (2.1) with $(x, y) = (v, z)$, we get $d_2(v)d_1(z) = -d_1(v)d_2(z)$ such that $-d_2(b)d_1(v)d_2(z)y + d_1(b)d_2(v)yd_2(z) = 0$. From equation (2.1) with $(x, y) = (b, v)$, we have $d_2(b)d_1(v) = -d_1(b)d_2(v)$ such that

$$\begin{aligned} d_1(b)d_2(v)d_2(z)y + d_1(b)d_2(v)yd_2(z) &= 0 \\ d_1(b)d_2(v)[d_2(z)y + yd_2(z)] &= 0. \end{aligned} \tag{2.8}$$

Since $d_2 \neq 0$, there exists $v_0 \in N$such that $d_2(v_0) \neq 0$. From equation (2.8) with $v = v_0$ we have $d_1(b)d_2(v_0)(d_2(z) \circ y) = 0$ for all $y, z \in N$. Suppose for the sake contradiction, that $d_1(b) \neq 0$. Since $d_2(v_0) \neq 0$ and N is prime such that $d_2(z)y + yd_2(z) = 0$ for all $y, z \in N$. With $y = d_2(z)$, so $2d_2(z)^2 = 0$. Since N is 2-torsion free, then $d_2(z)^2 = 0$. Now, replace y by $yd_2(z)$, then $d_2(z)yd_2(z) = -yd_2(z)^2$. So $d_2(z)yd_2(z) = 0$ for all $y, z \in N$. Since N is prime, for each fixed z, we conclude $d_2(z) = 0$ for all $z \in N$, i.e., $d_2 = 0$. This contradicts our assumption that $d_2 \neq 0$. Therefore, we must have $d_1(b) = 0$. From equation (2.6) with $d_1(b) = 0$, then $d_2(b)yd_1(z) = 0$ for all $y, z \in N$. Since $d_2(b) \neq 0$ and N is prime, weh have $d_1(z) = 0$ for all $z \in N$. Therefore, $d_1 = 0$.

Lemma 2.5. Let N be a prime near ring and h be a non zero homoderivation on N such that $h(N) \subseteq Z$. Then N is a commutative ring.

Proof. Since $h(N) \subseteq Z$, we have $h(x) \in Z$ for all $x \in N$. This means $h(x)$ commutes with every elemeny of N, then $h(x)y = yh(x)$ for all $x, y \in N$. Using the homoderivation definition $h(xy) = h(x)h(y) + h(x)y + xh(y)$. Since $h(x), h(y) \in Z$, we have $h(x)h(y) = h(y)h(x)$, $h(x)y = yh(x)$ and $xh(y) = h(y)x$. Therefore, $h(xy) = h(y)h(x) + h(y)x + yh(x)$. By definition of homoderivations, $h(yx) = h(y)h(x) + h(y)x + yh(x)$. Such that $h(xy) = h(yx)$ for all $x, y \in N$. Since h is additive and $h(xy) = h(yx)$, then $h([x, y]) = 0$ for all $x, y \in N$.

For any $x, y, z \in N$, consider $h([xy, z]) = 0$. Expand $[xy, z] = xyz - zxy$, then $h(xyz) - h(zxy) = 0$. Using the homoderivation definition and the fact that $h(x), h(y), h(z) \in Z$. After careful expansion and simplification, we obtain $h(x)[y, z] = h(y)[z, x]$. With cyclic permutation $(x, y, z) \mapsto (y, z, x)$, so $h(y)[z, x] = h(z)[x, y]$. Therefore, $h(x)[y, z] = h(y)[z, x] = h(z)[x, y]$. Since $h \neq 0$, there exists $x_0 \in N$ such that $h(x_0) \neq 0$. Note that $h(x_0) \in Z$. With $x = x_0$, then $h(x_0)[y, z] = h(y)[z, x_0] = h(y) \cdot 0 = 0$. So, $h(x_0)[y, z] = 0$ for all $y, z \in N$. Since $h(x_0) \neq 0, h(x_0) \in Z$ and N is prime, by the primeness condition $[y, z] = 0$ for all $y, z \in N$. Hence N is commutative.

Lemma 2.6. Let N be a prime near ring and h be a homoderivation on N. For any $z \in Z$ and $x \in N$, then $h(xz) = h(x)h(z) + h(x)z + xh(z)$ and $h(zx) = h(z)h(x) + h(z)x + zh(x)$. If $z \in Z$ and $h(z) \in Z$, then $h(xz) = h(x)z + xh(z) + h(x)h(z)$.

Proof. From definition of homoderivation, by substituting $y = z$ in the definition, for any $x, z \in N$, then $h(xz) = h(x)h(z) + h(x)z + xh(z)$. Similarly, with x replaced by z and y replaced by x, we have $h(zx) = h(z)h(x) + h(z)x + zh(x)$. Since $h(z) \in Z$, we have $h(x)h(z) = h(z)h(x)$ for all $x \in N$. Therefore, $h(xz) = h(x)z + xh(z) + h(x)h(z)$. When $z \in Z$ and $h(z) \in Z$, we have $zx = xz$, so $h(zx) = h(xz)$. With $h(z) \in Z$, we obtain $h(zx) = h(x)h(z) + xh(z) + zh(x)$. We get $h(x)h(z) + h(x)z + xh(z) = h(x)h(z) + xh(z) + zh(x)$ and sucht that $h(x)z = zh(x)$. Since $z \in Z$, this is automatically satisfied.

3. HOMODERIVATIONS AND ALGEBRAIC STRUCTURE OF PRIME NEAR RINGS

In this section, we presents the basic properties of homoderivations and the Leibniz rule, dependence theorems for homoderivations satisfying multiplicative identities and commutativity conditions obtained from Lie and Jordan product identities.

Theorem 3.1. *Let N be a prime near ring and h_1, h_2, h_3, h_4 be zero power valued non zero homoderivations on N. Suppose $h_1(x)h_2(y) = h_3(x)h_4(y)$ for all $x, y \in N$ if $h_{1|Z} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$.*

Proof. Let h_1, h_2, h_3, h_4 be zero power valued non zero homoderivations on N. Suppose that $h_1(x_1)h_2(x_2) = h_3(x_1)h_4(x_2)$ for all $x_1, x_2 \in N$. Since $h_{1|Z} \neq 0$, there is at least $0 \neq z \in Z$ such that $h_1(z) \neq 0$. It is clear that $h_1(z) \in Z$. Replacing x_1 by $x_1 z$ for $x_1 \in N$, we have

$$h_1(x_1 z)h_2(x_2) = h_3(x_1 z)h_4(x_2).$$

Thus

$$\begin{aligned} &h_1(x)h_1(z)h_2(x_2) + h_1(x_1)zh_2(x_2) + x_1h_1(z)h_2(x_2) \\ &= h_3(x_1)h_3(z)h_4(x_2) + h_3(x_1)zh_4(x_2) + x_1h_3(z)h_4(x_2) \end{aligned} \tag{3.1}$$

From Equation (3.1), for all $x_1, x_2 \in N$, the equation $h_1(x)h_1(z)h_2(x_2) = h_3(x_1)h_3(z)h_4(x_2)$ is obtained. In view of hypothesis, for all $x_2 \in N$, we get $h_1(z)h_2(x_2) = h_3(z)h_4(x_2)$ such that

$$\big(h_1(x_1) - h_3(x_1)\big)h_1(z)h_2(x_2) = 0 \text{ for all } x_1, x_2 \in N.$$

The primeness of N and $0 \neq h_1(z) \in Z$ imply that $\big(h_1(x_1) - h_3(x_1)\big)h_2(x_2) = 0$ for all $x_1, x_2 \in N$. Replacing x_2 by x_2x_3 such that $x_3 \in N$ obtained $\big(h_1(x_1) - h_3(x_1)\big)x_2h_2(x_3) = 0$ for all $x_1, x_2, x_3 \in N$. Since N is a prime near ring and h_2 is a non zero homoderivation of N, then $h_1(x_1) = h_3(x_1)$ for all $x_1 \in N$. In this case, by hypothesis, $h_1(x)h_2(x_2) = h_1(x_1)h_4(x_2)$ for all $x_1, x_2 \in N$. That is N. That is

$$h_1(x_1)\big(h_2(x_2) - h_4(x_2)\big) = 0 \text{ for all } x_1, x_2 \in N. \tag{3.2}$$

In equation (3.2), replacing x_1 by $x_1x_3 \in N$, we have

$$h_1(x_1)x_3\big(h_2(x_2) - h_4(x_2)\big) = 0 \text{ for all } x_1, x_2, x_3 \in N. \tag{3.3}$$

Since N is a prime near ring and h_1 is a non zero homoderivation of N, then $h_2(x_2) = h_4(x_2)$ for all $x_2 \in N$.

Theorem 3.2. *Let N be a prime near ring and h_1, h_2, h_3, h_4 be zero power valued non zero homoderivations on N. Suppose $h_1(x)h_2(y) = h_3(x)h_4(y)$ for all $x, y \in N$, then the following hold:*
(i) $h_{1|z} = 0$ if and only if $h_{3|z} = 0$,
(ii) if there exists $z_0 \in Z$ such that $h_2(z_0) = h_4(z_0) \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$,
(iii) $h_{2|z} = 0$ if and only if $h_{4|z} = 0$.

Proof. (i) (⇒) Based on assumptions $h_1(x)h_2(y) = h_3(x)h_4(y)$, replace x by $z \in Z$, obtained $h_1(z)h_2(y) = h_3(z)h_4(y)$ for all $y \in N$. Since $h_{1|z} = 0$, we have $h_1(z) = 0$ for all $z \in Z$. Thus

$$h_3(z)h_4(y) = 0 \text{ for all } y \in N. \tag{3.4}$$

In equation (3.4), replace y by yw for arbitrary $w \in N$, we get $h_3(z)h_4(yw) = 0$. Using the definition of homoderivation $h_3(z)\big(h_4(y)h_4(w) + h_4(y)w + yh_4(w)\big) = 0$.

Since $h_3(z)h_4(y) = 0$ and $h_3(z)h_4(w) = 0$, the first two terms vanish $h_3(z)yh_4(w) = 0$ for a ll $y, w \in N$. Since N is a prime near ring and h_4 is non zero, there exists $w_0 \in N$ suc that $h_4(w_0) \neq 0$. With $w = w_0$, we have $h_3(z)yh_4(w_0) = 0$ for all $y \in N$. Since N is prime and $h_4(w_0) \neq 0$, then $h_3(z) = 0$. Since $z \in Z$ was arbitrary $h_{3|z} = 0$.

(⇐) Since $h_{3|z} = 0$, we have $h_3(z) = 0$ for all $z \in Z$. Based on assumptions $h_1(x)h_2(y) = h_3(x)h_4(y)$, replace x by $z \in Z$, we get $h_1(z)h_2(y) = h_3(z)h_4(y) = 0$ for all $y \in N$. Thus

$$h_1(z)h_2(y) = 0 \text{ for all } y \in N. \tag{3.5}$$

Replace y by yw for arbitrary $w \in N$, $h_1(z)h_2(yw) = 0$. Using the definition of homoderivation for h_2, we obtained $h_1(z)\big(h_2(y)h_2(w) + h_2(y)w + yh_2(w)\big) = 0$. Because $h_1(z)h_2(y) = 0$ and $h_1(z)h_2(w) = 0$, so the first two terms vanish $h_1(z)yh_2(w) = 0$ for all $y, w \in N$.

Since N is prime and h_2 is non zero, there exists $w_0 \in N$ such that $h_2(w_0) \neq 0$. With $w = w_0$, we have $h_1(z)yh_2(w_0) = 0$ for all $y \in N$. Since N is prime and $h_2(w_0) \neq 0$, then $h_1(z) = 0$. Since $z \in Z$ was arbitrary $h_{1|z} = 0$.

(ii) Assume that there exists $z_0 \in Z$ such that $h_2(z_0) = h_4(z) \neq 0$. Substituting $y = z_0$ into $h_1(x)h_2(z_0) = h_3(x)h_4(z_0)$. Since $h_2(z_0) = h_4(z_0)$, it follows that $\big(h_1(x) - h_3(x)\big)h_2(z_0) = 0$ for every $x \in N$. Because $h_2(z_0) \neq 0$ and N is prime, we conclude that $h_1(x) = h_3(x)$ for all $x \in N$. Hence, $h_1 = h_3$. Substituting this equality into the hypothesis $h_1(x)h_2(y) = h_3(x)h_4(y)$, gives $h_1(x)\big(h_2(y) - h_4(y)\big) = 0$ for all $x, y \in N$. Since h_1 is nonzero, there exists $x_0 \in N$ such that $h_1(x_0) \neq 0$. Thus $h_1(x_0)\big(h_2(y) - h_4(y)\big) = 0$ for every $y \in N$. By the primeness of N, $h_2(y) = h_4(y)$ for every $y \in N$. Therefore $h_2 = h_4$. Hence, $h_1 = h_3$ and $h_2 = h_4$.

(iii)(⇒) Assume that $h_{2|z} = 0$, we have $h_2(z) = 0$ for all $z \in Z$. Based on assumption $h_1(x)h_2(y) = h_3(x)h_4(y)$, replace y by $z \in Z$, then $h_1(x)h_2(z) = h_3(x)h_4(z)$ for all $x \in N$. Since $h_2(z) = 0$, then $0 = h_3(x)h_4(z)$ for all $x \in N$. Thus $h_3(x)h_4(z) = 0$ for all $x \in N$. Since h_3 is non zero, there exists $x_0 \in N$ such that $h_3(x_0) \neq 0$. With $x = x_0$, then $h_3(x_0)h_4(z) = 0$. Since N is prime and $h_3(x_0) \neq 0$, we get $h_4(z) = 0$. Since $z \in Z$ was arbitrary $h_{4|z} = 0$.

(⇐) Since $h_{4|z} = 0$, we have $h_4(z) = 0$ for all $z \in Z$. Based on assumption $h_1(x)h_2(y) = h_3(x)h_4(y)$, replace y by $z \in Z$, then $h_1(x)h_2(z) = h_3(x)h_4(z) = 0$ for all $x \in N$. Thus $h_1(x)h_2(z) = 0$ for $x \in N$. Since h_1 is non zero, there exist $x_0 \in N$ such that $h_1(x_0) \neq 0$. With $x = x_0$, we obtained $h_1(x_0)h_2(z) = 0$. Since N is prime and $h_1(x_0) \neq 0$, then $h_2(z) = 0$. Since $z \in Z$ was arbitrary $h_{2|z} = 0$.

Theorem 3.3. *Let N be a prime near ring and h_1, h_2 be zero power valued non zero homoderivations on N. Suppose there exist $a, b \in N$ such that $ah_1(x) + h_2(x)b = 0$ for all $x \in N$ then $a = -b \in Z$ or $h_{1|z} = h_{2|z} = 0$.*

Proof. Let h_1 and h_2 be zero power valued non zero homoderivations on N. Suppose that there are $a, b \in N$ such that $ah_1(x) + h_2(x)b = 0$ for all $x \in N$. If $a = b = 0$, then the proof is clear. From now, $a \neq 0$ and $b \neq 0$. Suppose that $h_{1|z} = 0$. Replacing x by z for $z \in Z$, then $ah_1(z) + h_2(z)b = 0$. Since $h_{1|z} = 0$ then $h_2(z)b = 0$. This means that $h_2(z) = 0$ for all $z \in Z$, by the primeness of N. With the same arguments above, it can be shown that if $h_{2|z} = 0$ then $h_{1|z} = 0$.

Assume that $h_{1|z} \neq 0$. Because $h_{1|z} \neq 0$, there is at least $0 \neq z_1 \in Z$ such that $0 \neq h_1(z_1) \in Z$ and $h_2(z_1) \in Z$. Replacing x by xz_1, we have $0 = ah_1(xz_1) + h_2(xz_1)b$. Using the definition of homoderivation

$$h_1(xz_1) = h_1(x)h_1(z_1) + h_1(x)z_1 + xh_1(z_1)$$
$$h_2(xz_1) = h_2(x)h_2(z_1) + h_2(x)z_1 + xh_2(z_1).$$

Thus, $0 = a\big(h_1(x)h_1(z_1) + h_1(x)z_1 + xh_1(z_1)\big) + \big(h_2(x)h_2(z_1) + h_2(x)z_1 + xh_2(z_1)\big)b$. Using $z_1, h_1(z_1), h_2(z_1) \in Z$, obtained $(a(h_1(x) + x) - (h_2(x) + x)a)h_1(z_1) = 0$.

Since N is a prime near ring and $h_1(z_1) \neq 0$, for all $x \in N$, then $a(h_1(x) + x) - (h_2(x) + x)a = 0$. Since $h_{2|z} \neq 0$, there is at least $0 \neq z_2 \in Z$ such that $0 \neq h_2(z_2) \in Z$. Replacing x by z_2, we get $ah_1(z_2) + h_2(z_2)(-a) = 0$. Combinig the last equations and $ah_1(x) + h_2(x)b = 0$, thus $h_2(z_2)(b + a) = 0$. The primeness of N and $h_2(z_2) \neq 0$ implies $a = -b$. In that case, for any $x \in N$, then $ah_1(x) - h_2(x)a = 0$. Replacing x by xz_1 such that $0 = ah_1(xz_1) - h_2(xz_1)a$. Expanding

$$0 = a(h_1(x)h_1(z_1) + h_1(x)z_1 + xh_1z_1)) - (h_2(x)h_2(z_1) + h_2(x)z_1 + xh_2(z_1))a.$$

According to the last equation, we have

$$ah_1(x)h_1(z_1) + axh_1(z_1) - ah_1(x)h_1(z_1) - xah_1(z_1) = 0.$$

This implies $[a, x]h_1(z_1) = 0$ for all $x \in N$. The primeness of $N$and $h_1(z_1) \neq 0$ implies $a \in Z$.

Lemma 3.4. *Let N be a 2-torsion free prime near ring. Let $u, c \in N$ such that $u \neq 0$ and $[u, c] = 0$. If $uc + cu = 0$ then $c = 0$.*

Proof. Since $[u, c] = 0$, we have $uc = cu$. By the assumption $uc + cu = 0$. Substituting $cu = uc$, we obtain $uc + cu = 0$. Which implies $2uc = 0$. Since N is 2-torsion free, it follows that $uc = 0$. Because N is a prime near ring and $u \neq 0$, we conclude that $c = 0$. Therefore, $c = 0$.

Theorem 3.5. *Let N be a 2-torsion free prime near ring. Suppose h is a non zero homoderivation on N such that $h([x, y]) = 0$ for all $x, y \in N$. Then N is a commutative ring.*

Proof. In view of our hypothesis, we have $h(xy) = h(yx)$ for all $x, y \in N$. Using the definition of homoderivation, can be written as

$$h(x)h(y) + h(x)y + xh(y) = h(y)h(x) + h(y)x + yh(x)$$

for all $x, y \in N$. Rearranging $h(x)y + xh(y) - h(y)x - yh(x) = h(y)h(x) - h(x)h(y)$ for all $x, y \in N$. Replace y by $[y, z]$ and since $h([y, z]) = 0$ for all $y, z \in N$, we obtain $\big[h(x), [y, z]\big] = 0$. Now, replace x by xy, y by z, z by t, and using definition of homoderivation and the commutator identity, we get $h(x)\big[y, [z, t]\big] + \big[x, [z, t]\big]h(y) = 0$. Because $h \neq 0$, there exists $a \in N$ such that $h(a) \neq 0$.

Taking $x = a$, we have

$$h(a)\big[y, [z, t]\big] + \big[a, [z, t]\big]h(y) = 0. \tag{3.6}$$

Replacing y by $y + a$, we obtain $h(a)\big[y + a, [z, t]\big] + \big[a, [z, t]\big]h(y + a) = 0$. Since h is additive, so

$$h(a)\big[y, [z, t]\big] + h(a)\big[a, [z, t]\big] + \big[a, [z, t]\big]h(y) + \big[a, [z, t]\big]h(a) = 0$$
$$h(a)\big[a, [z, t]\big] + \big[a, [z, t]\big]h(a) = 0.$$

Applying Lemma 3.4, we obtain $\left[a,[z,t]\right]=0$. Substituting $\left[a,[z,t]\right]=0$ to equation (3.6), we get $h(a)\left[y,[z,t]\right]=0$. Since N is prime and $h(a)\neq 0$, it follows that $\left[y,[z,t]\right]=0$ for all $y,z,t\in N$. Hence every commutator belongs to the center of N. By the standard result for 2-torsion free prime near rings, then $[z,t]=0$ for all $z,t\in N$. Therefore $zt=tz$ for all $z,t\in N$ and hence N is a commutative ring.

Theorem 3.6. *Let N be a 2-torsion free prime near ring. Suppose h is a non zero homoderivation on N such that $h(x\circ y)=0$ for all $x,y\in N$, then N is a commutative ring.*

Proof. By definition of Jordan product $x\circ y=xy+yx$ and the hypothesis $h(x\circ y)=0$ implies $h(xy+yx)=0$. Since h is additive, then $h(xy)+h(yx)=0$. Thus

$$h(xy)=-h(yx) \text{ for all} x,y\in N. \tag{3.7}$$

Using the definition about homoderivations, we have

$$h(x)h(y)+h(x)y+xh(y)=-(h(y)h(x)+h(y)x+yh(x))$$
$$h(x)h(y)+h(x)y+xh(y)+h(y)h(x)+h(y)x+yh(x)=0. \tag{3.8}$$

Replace x by yx in Equation (3.7), we get $h\big((yx)y\big)=-h(y(yx))$. That is $h(yxy)=-h(y^2x)$. Using the definition of homoderivation, we obtaine

$h(yxy)$
$=h(yx)h(y)+h(yx)y+(yx)h(y)$
$=\big(h(y)h(x)+h(y)x+yh(x)\big)h(y)+\big(h(y)h(x)+h(y)x+yh(x)\big)y+yxh(y)$
$=h(y)h(x)h(y)+h(y)xh(y)+yh(x)h(y)+h(y)h(x)y+h(y)xy+yh(x)y+yxh(y).$

For the right side, using the definition of homoderivation, we have

$h(y^2x)$
$=h(y^2)h(x)+h(y^2)x+y^2h(x)$
$=\big(h(y)^2+h(y)y+yh(y)\big)h(x)+\big(h(y)^2+h(y)y+yh(y)\big)x+y^2h(x)$
$=h(y)^2h(x)+h(y)yh(x)+yh(y)h(x)+h(y)^2x+h(y)yx+yh(y)x+y^2h(x).$

Therefore,

$$h(y)h(x)h(y)+h(y)xh(y)+yh(x)h(y)+h(y)h(x)y+h(y)xy+yh(x)y+yxh(y) \tag{3.9}$$
$$=-(h(y)^2h(x)+h(y)yh(x)+yh(y)h(x)+h(y)^2x+h(y)yx+yh(y)x+y^2h(x)).$$

Setting $x=y$ in Equation (3.8), we obtaine $2\big(h(y)^2+h(y)y+yh(y)\big)=0$. Since N is 2-torsion free, this implies

$$h(y)^2+h(y)y+yh(y)=0 \text{ for all } y\in N. \tag{3.10}$$

Consequently, $h(y^2)=0$ for all $y\in N$. Using Equation (3.10) to simplify Equation (3.9), we get $h(y)h(x)h(y)+h(y)xh(y)+yh(x)h(y)+h(y)h(x)y+h(y)xy+yh(x)y+yxh(y)=-y^2h(x)$. Replacing x by xz and using the definition of homoderivation, we obtain after simplication

$$h(y)xh(y)z=h(y)zxh(y) \text{ for all } x,y,z\in N. \tag{3.11}$$

Using Equation (3.9) and Equation (3.11), we have $h(y)x\big(-h(y)y-yh(y)\big)=h(y)zxh(y)$. This implies $h(y)\big(xh(y)y+xyh(y)+zxh(y)\big)=0$. Replacing z by $z+y$ in Equation (3.11), we obtain $h(y)[z,h(y)]x=0$ for all $x,y,z\in N$. Since N is prime, yields that for each fixed $y\in N$, either $h(y)=0$ or $[z,h(y)]=0$ for all $z\in N$. In the latter case, $h(y)\in Z$. Therefore, we conclude that $h(y)\in Z$ for all $y\in N$, that is $h(N)\subseteq Z$. Thus N is a commutative ring.

Theorem 3.7. *Let L be a non zero Lie ideal of N and h be a non zero homoderivation of N such that $h(x)=0$ for all $x\in L$. Then $L\subseteq Z$.*

Proof. Let L be a non zero Lie ideal of N and h be a non zero homoderivation of N such that $h(x)=0$ for all $x\in L$. Since h is homoderivation of N, we have

$$h([x_1,r_1])=[h(x_1),h(r_1)]+[h(x_1),r_1]+[x_1,h(r_1)] \text{ for all } x_1\in L, r_1\in N.$$

By hypothesis, $[x_1, h(r_1)] = 0$ for all $x_1 \in L$ and $r_1 \in N$. By taking $r_1 = r_1 x_2$, for any $x_2 \in L$, in the last equation $h(r_1)[x_1, x_2] = 0$ for all $x_1, x_2 \in L, r_1 \in N$. Replacing r_1 by $r_1 r_2, r_2 \in N$, we get $h(r_1) r_2 [x_1, x_2] = 0$. Hence $[x_1, x_2] = 0$ for all $x_1, x_2 \in L$, by the primeness of N. By replacing x_2 by $[x_2, r_1]$ in the last equation $\left[x_1, [x_2, r_1]\right] = 0$ for all $x_1, x_2 \in L, r_1 \in N$.

Consider two inner derivations of $N, I_{x_1}: N \to N$ and $I_{x_2}: N \to N$ defined by $I_{x_1}(s) = [x_1, s]$ and $I_{x_2}(x) = [x_2, s]$. Thus, $I_{x_1} I_{x_2}(r_1) = 0$ for all $r_1 \in N$. Therefore, $I_{x_1} = 0$ or $I_{x_2} = 0$. That is, $x_1 \in Z$ or $x_2 \in Z$. This proves that $L \subseteq Z$.

Theorem 3.8. *Let N be a 2-torsion free prime near ring. Let L be a non zero square closed Lie ideal of N and h be a non zero homoderivation of N such that $h(x) \in Z$ for all $x \in L$. Then $L \subseteq Z$.*

Proof. By hypothesis, we have $h(x) \in Z(N)$ for all $x \in L$. For all $x_1 \in L$ and $r_1 \in N$, we get $[h(x_1), r_1] = 0$. Replacing x_1 by x_1^2, we get $[h(x_1^2), r_1] = 0$. Expanding $h(x_1^2)$ by using the definition of homoderivation, we get $h(x_1^2) = h(x_1)^2 + h(x_1)x_1 + x_1 h(x_1)$. In view of the fact that $h(x_1) \in Z(N)$, we compute we compute

$$\begin{aligned}
[h(x_1)^2, r_1] &= [h(x_1)^2 + h(x_1)x_1 + x_1 h(x_1), r_1] \\
&= [h(x_1)^2, r_1] + [h(x_1)x_1, r_1] + [x_1 h(x_1), r_1] \\
&= 0 + h(x_1)[x_1, r_1] + [h(x_1), r_1]x_1 + x_1[h(x_1), r_1] + [x_1, r_1]h(x_1) \\
&= 2h(x_1)[x_1, r_1].
\end{aligned}$$

Therefore, we have that $2h(x_1)[x_1, r_1] = 0$. Since N is 2-torsion free, we arrive at $h(x_1)[x_1, r_1] = 0$ for all $x_1 \in L, r_1 \in N$. Substituting $r_1 r_2$ instead of $r_1, r_2 \in N$, then $h(x_1) r_1 [x_1, r_2] = 0$.

The primeness of N implies $h(x_1) = 0$ or $x_1 \in Z$, for all $x_1 \in L$. Define $A = \{x \in L: h(x) = 0\}$ and $B = \{x \in L: x \in Z\}$. Note that both are additive subgroups of L and their union equals L. Thus, either $A = L$ or $B = L$. Suppose first that $A = L$, then, $h(L) = \{0\}$. In view of Theorem 3.7, $L \subseteq Z$. In other case, $x_1 \in Z$, for all $x_1 \in L$. That is $L \subseteq Z$.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

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