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L(3,1)-Labeling of Double-Quadrilateral Windmill And Flower Graphs

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Abstract

The $L(3{,}1)$-labeling of a graph is a distance-constrained labeling in which adjacent vertices receive labels differing by at least 3, while vertices at distance two receive labels differing by at least 1. The span of an $L(3{,}1)$-labeling is defined as the largest label assigned to any vertex of the graph G. The minimum span among all such labelings, denoted by $\lambda_{3,1}(G)$, is called the $L(3{,}1)$-labeling number of G. In this paper, we determine the exact $L(3{,}1)$-labeling numbers of double-quadrilateral windmill graphs and double-quadrilateral flower graphs. By constructing feasible labelings and establishing sharp lower bounds, we prove that $\lambda_{3,1}(DQ_k) = 3k + 2$ and $\lambda_{3,1}(FDQ_k) = 2k + 3$ for every integer $k \geq 2$. These results provide the first $L(3{,}1)$-labeling characterization of these graph families and contribute to the study of distance-constrained graph labeling on graphs composed of interconnected quadrilateral cycles.

Keywords: $L(3{,}1)$-labeling, double-quadrilateral windmill graph, double-quadrilateral flower graph.

1. INTRODUCTION AND PRELIMINARIES

Graph labeling is one of the rapidly developing areas of graph theory and continues to attract significant research attention because of its rich theoretical foundations and wide range of practical applications [11]. In general, graph labeling is a function that assigns integer labels to the vertices, edges, or both, subject to specified conditions [5]. Various labeling schemes have been introduced, including graceful labeling [17], harmonious labeling [7], prime labeling [18], odd prime labeling [12], total k-labeling [19], and distance-constrained labeling [8]. Beyond its theoretical importance, graph labeling has been applied in communication networks [16], crystallography [13], dental arch analysis [14], chemistry [2], and many other disciplines.

One of the most extensively studied distance-constrained labelings is the $L(h,k)$-labeling. Let $G = (V, E)$ be a connected, simple, and undirected graph. An $L(h,k)$-labeling of G is a function $f: V(G) \to \{0,1,\dots,\mathbb{N}\}$ such that, for every pair of vertices $u, v \in V(G)$, $|f(u) - f(v)| \geq h$ whenever $d(u,v) = 1$, and $|f(u) - f(v)| \geq k$ whenever $d(u,v) = 2$. The maximum label assigned by an $L(h,k)$-labeling f of G is called the span of the labeling and is denoted by $span(f)$. Since a graph G may admit multiple $L(h,k)$-labelings with different spans, the $L(h,k)$-labeling number of G is defined as the minimum span among all $L(h,k)$-labelings of G, and is denoted by $\lambda_{h,k}(G)$ [8]. This concept was originally motivated by the channel assignment problem in wireless communication networks, where nearby transmitters must be assigned sufficiently separated frequencies to avoid interference.

A special case that has attracted considerable attention is the $L(3,1)$-labeling, in which adjacent vertices must receive labels that differ by at least 3, while vertices at distance two must receive labels that differ by at least 1 [6]. Compared with other distance-constrained labeling schemes, $L(3,1)$-labeling imposes stricter constraints on adjacent vertices, making the labeling problem more challenging. Consequently, determining the minimum span of an $L(3,1)$-labeling has become an active area of research in graph labeling theory. Several studies have established exact values or bounds for the $L(3,1)$-labeling numbers of various graph classes, including paths, cycles, complete graphs, complete bipartite graphs, stars, and bistars [6], supercycle graphs [3], comb products of star and cycle graphs [1], pendulum graphs and cogon grass graphs [9], as well as several classes of star-related graphs [10]. These results demonstrate that the structural properties of a graph play a crucial role in determining its $L(3,1)$-labeling number and motivate further investigations into more complex graph families.

Although numerous results on $L(3,1)$-labeling have been established for various graph classes, studies on graph families constructed from combinations of multiple cycles remain relatively limited. In particular, to the best of the authors' knowledge, no previous study has investigated the $L(3,1)$-labeling problem on double-quadrilateral windmill graphs or double-quadrilateral flower graphs. These graph families possess more complex structures than those considered in earlier studies because they contain multiple quadrilateral cycles connected through specific vertices, resulting in more intricate adjacency and distance relationships. Consequently, labeling techniques developed for other graph classes cannot be applied directly, and new labeling constructions are therefore required. Motivated by this research gap, we investigate the $L(3,1)$-labeling problem on double-quadrilateral windmill graphs and double-quadrilateral flower graphs. Specifically, we determine the exact values of the corresponding $L(3,1)$-labeling numbers by constructing feasible labelings and establishing sharp lower bounds. Our results extend the existing theory of distance-constrained graph labeling and provide new insights into labeling problems on graph families composed of interconnected quadrilateral cycles.

2. PRELIMINARIES

We now introduce the specific graph families that are the main focus of this paper, along with the definitions and lemmas needed for the subsequent results.

Definition 2.1. *A double-quadrilateral graph, denoted by DQ, is a graph obtained from the cycle graph C_6 by adding an edge between a pair of vertices whose distance in C_6 is three [5].*

Definition 2.2. *A double-quadrilateral windmill graph, denoted by DQ_k, is constructed from k copies of the double-quadrilateral graph by identifying one vertex of degree three from each copy into a single common vertex [4]. The vertex set of DQ_k is given by $V(DQ_k) = \{v_0\} \cup \{v_j^i \mid j = 1,2,3 \; and \; i = 1,2,,\dots,k\} \cup \{w_l^i \mid l = 1,2 \; and \; i = 1,2,\dots,k\}$, and its edge set is given by $E(DQ_k) = \{v_0 v_j^i \mid j = 1,2,3 \; and \; i = 1,2,\dots,k\} \cup \{v_1^i w_1^i, w_1^i v_3^i, v_3^i w_2^i, w_2^i v_2^i \mid i = 1,2,\dots,k\}$.*

Definition 2.3. *A double-quadrilateral flower graph, denoted by FDQ_k,, is obtained from the double-quadrilateral windmill graph by joining one edge between each pair of consecutive neighboring double-quadrilateral subgraphs [15]. The vertex set of FDQ_k is given by $V(FDQ_K) =$*

$\{v_0\} \cup \{v_i | i = 1, 2, \dots, 2k+1\} \cup \{w_j^i | j = 1, 2 \textit{ and } i = 1, 2, \dots, k\}$, *and the edge set is given by* $E(FDQ_k) = \{v_0 v_i | i = 1, 2, \dots, 2k+1\} \cup \{v_{2i-1} w_1^i | i = 1, 2, \dots, k\} \cup \{w_1^i v_{2i} | i = 1, 2, \dots, k\} \cup \{v_{2i} w_2^i | i = 1, 2, \dots, k\} \cup \{w_2^i v_{2i+1} | i = 1, 2, \dots, k\}$.

Lemma 2.1. *If H is a subgraph of a graph G, then $\lambda_{3,1}(H) \leq \lambda_{3,1}(G)$ [6].*

Lemma 2.2. *For every positive integer n, $\lambda_{3,1}(S_n) = n + 2$ [6].*

The vertex and edge notation of (DQ_k) and (FDQ_k) is illustrated in Figures 2.1 and 2.2, respectively.

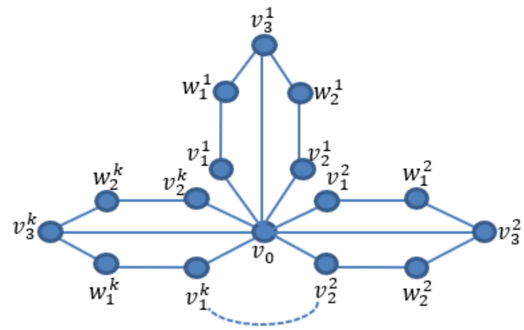

Figure 2.1. The Double-Quadrilateral Windmill Graph DQ_k

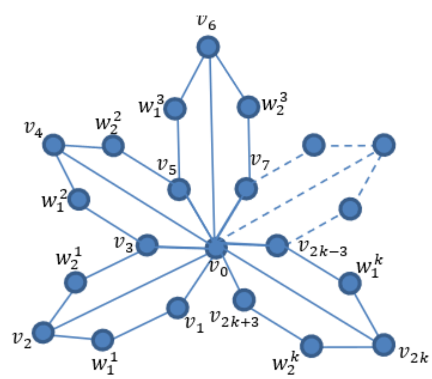

Figure 2.2. The Double-Quadrilateral Flower Graph FDQ_k

3. MAIN RESULTS

In this section, we determine the $L(3,1)$-labeling numbers of the double-quadrilateral windmill graphs DQ_k and the double-quadrilateral flower graphs FDQ_k, whose structures are defined in Section 2. The exact values of $\lambda_{3,1}(DQ_k)$ and $\lambda_{3,1}(FDQ_k)$ are established by proving the corresponding lower and upper bounds. The lower bounds are obtained by applying Lemmas 2.1 and 2.2 to suitable star subgraphs, while the upper bounds are established through explicit $L(3,1)$-labeling constructions.

Theorem 3.1. *For every integer $k \geq 2$, the double-quadrilateral windmill graph DQ_k satisfies $\lambda_{3,1}(DQ_k) = 3k + 2$.*

Proof. To prove that $\lambda_{3,1}(DQ_k) = 3k + 2$, it suffices to show that $\lambda_{3,1}(DQ_k) \geq 3k + 2$ and $\lambda_{3,1}(DQ_k) \leq 3k + 2$. First, we establish the lower bound. Observe that the star graph S_{3k} is a subgraph of the double-quadrilateral windmill graph DQ_k. By Lemma 1.1 and Lemma 1.2, we obtain $\lambda_{3,1}(DQ_k) \geq \lambda_{3,1}(S_{3k}) = 3k + 2$. Hence, $\lambda_{3,1}(DQ_k) \geq 3k + 2$. Next, we establish the upper bound by constructing an $L(3,1)$-labeling of DQ_k with span $3k + 2$. Define a function $f: V(DQ_k) \to \{0, 1, \dots, 3k + 2\}$ as follows.

$$f(v_0) = 0. \tag{3.1}$$

For $i = 1,2, \dots, k$.

$$f(v_j^i) = \begin{cases} 3i, & \text{for } j = 1, \\ 3i + 1, & \text{for } j = 2, \\ 3i + 2, & \text{for } j = 3. \end{cases} \tag{3.2}$$

$$f(w_j^1) = \begin{cases} 8, & \text{for } j = 1, \\ 1, & \text{for } j = 2. \end{cases} \tag{3.3}$$

For $i = 2, 3, \dots, k$

$$f(w_j^i) = \begin{cases} 1, & \text{for } j = 1, \\ 2, & \text{for } j = 2. \end{cases} \tag{3.4}$$

Next, it will be shown that every pair of adjacent vertices receives labels differing by at least 3. Since v_0 is adjacent to v_j^i, we have $|f(v_0) - f(v_1^i)| = 3i \geq 3$, $|f(v_0) - f(v_2^i)| = 3i + 1 \geq 3$, and $|f(v_0) - f(v_3^i)| = 3i + 2 \geq 3$, for every $i = 1,2, \dots, k$. Furthermore, $|f(v_1^1) - f(w_1^1)| = |3 - 8| = 5 \geq 3$, $|f(v_3^1) - f(w_1^1)| = |5 - 8| = 3 \geq 3$, $|f(v_2^1) - f(w_2^1)| = |4 - 1| = 4 \geq 3$, and $|f(v_3^1) - f(w_2^1)| = |5 - 1| = 4 \geq 3$. For $i = 2,3, \dots, k$, we obtain $|f(v_1^i) - f(w_1^i)| = |3i - 1| \geq 3$, $|f(v_3^i) - f(w_1^i)| = |(3i + 1) - 1| \geq 3$, $|f(v_2^i) - f(w_2^i)| = |(3i + 1) - 2| \geq 3$, and $|f(v_3^i) - f(w_2^i)| = |(3i + 2) - 2| \geq 3$. Therefore, every pair of adjacent vertices receives labels differing by at least 3. It remains to verify the condition for vertices at distance two. Observe that, for every $i = 1,2, \dots, k$ and $j = 1,2,3$, the labels assigned to the vertices v_j^i are distinct. Likewise, the labels assigned to v_0, w_1^i, and w_2^i are all distinct. In addition, $f(w_1^i) \neq f(w_2^i)$ for every $i = 1,2, \dots, k$. Hence, any two vertices at distance two receive different labels, and therefore their label difference is at least 1. Consequently, the function f satisfies the conditions of an $L(3,1)$-labeling of DQ_k with span $3k + 2$. Therefore, $\lambda_{3,1}(DQ_k) \leq 3k + 2$. Combining this result with the previously established lower bound $\lambda_{3,1}(DQ_k) \geq 3k + 2$, we conclude that $\lambda_{3,1}(DQ_k) = 3k + 2$. ■

To illustrate the $L(3,1)$-labeling construction described in Theorem 3.1, we provide an example of the labeling of DQ_k in Figure 3.1. The figure illustrates the labeling pattern obtained from the construction in the proof and demonstrates that the labeling satisfies the required distance constraints.

Having established the $L(3,1)$-labeling number of the double-quadrilateral windmill graph DQ_k and illustrated the corresponding labeling construction in Figure 3.1, we now turn our attention to the double-quadrilateral flower graph FDQ_k. The following theorem determines its exact (L(3,1))-labeling number.

Theorem 3.2. For every integer $k \geq 2$, the double-quadrilateral flower graph FDQ_k satisfies $\lambda_{3,1}(FDQ_k) = 2k + 3$.

Proof. To establish that $\lambda_{3,1}(FDQ_k) = 2k + 3$, it is sufficient to prove both $\lambda_{3,1}(FDQ_k) \geq 2k + 3$ (3.5) and $\lambda_{3,1}(FDQ_k) \leq 2k + 3$. We begin by proving the lower bound. Since the star graph S_{2k+1} is a (3.6) subgraph of the double-quadrilateral flower graph FDQ_k, Lemma 2.1 together with Lemma 2.2 (3.7) implies that $\lambda_{3,1}(FDQ_k) \geq \lambda_{3,1}(S_{2k+1}) = 2k + 3$. Therefore, $\lambda_{3,1}(FDQ_k) \geq 2k + 3$. To complete the proof, it remains to establish the upper bound. This is achieved by constructing an $L(3,1)$-labeling of FDQ_k whose span is $2k + 3$. Define the function $f: V(FDQ_k) \to \{0,1, \dots, 2k + 3\}$ as follows.

$$f(v_0) = 0$$

$$f(v_i) = i + 2, \text{ for } i = 1, 2, \dots, 2k + 1$$

$$f(w_j^1) = \begin{cases} 7, & \text{for } j = 1, \\ 1, & \text{for } j = 2. \end{cases}$$

For $i = 2, 3, \dots, k$.

$$f(w_j^i) = \begin{cases} 2, & \text{for } j = 1, \\ 1, & \text{for } j = 2. \end{cases} \tag{3.8}$$

We now verify that f satisfies the conditions of an $L(3,1)$-labeling. First, v_0 is adjacent to every v_i, where $i = 1,2,\dots,2k+1$. Thus, $|f(v_0) - f(v_i)| = |0 - (i+2)| = i + 2 \geq 3$ Next, any two distinct vertices v_i and v_j are at distance two through v_0. Since $f(v_i) = i + 2$ for $i = 1,2,\dots,2k+1$, the labels assigned to $v_1, v_2, \dots, v_{2k+1}$ are distinct. Hence, $\left|f(v_i) - f(v_j)\right| \geq 1$ for $i \neq j$. For the edges involving w_1^1 and w_2^1, we have $|f(v_1) - f(w_1^1)| = |3 - 7| = 4 \geq 3$, $|f(v_2) - f(w_2^1)| = |4 - 1| = 3 \geq 3$, and $|f(v_3) - f(w_2^1)| = |5 - 1| = 4 \geq 3$. For $i = 2,3,\dots,k$, the vertex v_{2i} is adjacent to both w_1^i and w_2^i. Therefore, $|f(v_{2i}) - f(w_1^i)| = |(2i+2) - 2| = 2i \geq 3$ and $|f(v_{2i}) - f(w_1^i)| = |(2i+2) - 1| = 2i + 1 \geq 3$. Furthermore, for $i = 2,3 \dots, k$, the vertex v_{2i-1} is adjacent to w_1^i. For $i = 2,3 \dots, k$, $|f(v_{2i-1}) - f(w_1^i)| = |(2i+1) - 2| = 2i - 1 \geq 3$. Similarly, v_{2i+1} is adjacent to w_2^i. For $i = 2,3 \dots, k$, $|f(v_{2i+1}) - f(w_2^i)| = |2i + 3 - 1| = 2i + 2 \geq 3$. Thus, every pair of adjacent vertices receives labels differing by at least 3, and every pair of vertices at distance two receives labels differing by at least 1. Hence, f is an $L(3,1)$-labeling of FDQ_k. Since the largest label assigned by f is $2k+3$, we obtain $\lambda_{3,1}(FDQ_k) \leq 2k + 3$. Combining the lower and upper bounds, we conclude that $\lambda_{3,1}(FDQ_k) = 2k + 3$. ■

To illustrate the labeling construction in Theorem 3.2, we consider the case $k = 3$. Figure 3.2 presents an $L(3,1)$-labeling of FDQ_3 obtained from the labeling function defined in the proof of Theorem 3.2. This example illustrates the labeling pattern used to establish the upper bound $(\lambda_{3,1}(FDQ_k) \leq\ 2k + 3$.

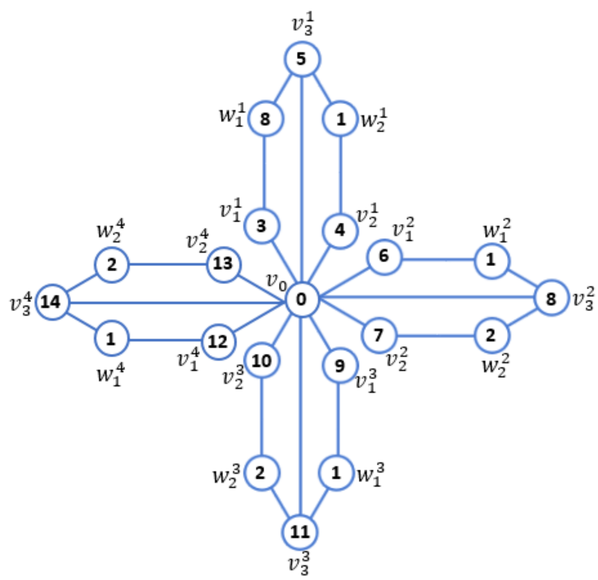

Figure 3.1. An $L(3,1)$-labeling of DQ_4.

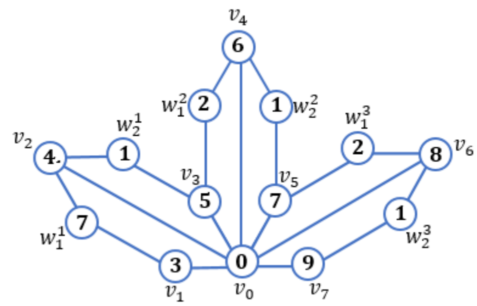

Figure 3.2. An $L(3{,}1)$-labeling of FDQ_3.

The results presented in this section establish the exact $L(3{,}1)$-labeling numbers of the double-quadrilateral windmill graph DQ_k and the double-quadrilateral flower graph FDQ_k. The corresponding labeling constructions and their illustrative examples confirm the attainability of the established bounds. Thus, the exact values of $\lambda_{3,1}(DQ_k)$ and $\lambda_{3,1}(FDQ_k)$ are completely determined for every $k \geq 2$.

4. CONCLUSION

In this paper, we investigated the $L(3{,}1)$-labeling of two graph families constructed from interconnected double-quadrilateral subgraphs, namely the double-quadrilateral windmill graph DQ_k and the double-quadrilateral flower graph FDQ_k. By establishing suitable lower bounds through star subgraphs and constructing explicit $L(3{,}1)$-labelings, we determined their exact $L(3{,}1)$-labeling numbers. For every integer $k \geq 2$, we obtained $\lambda_{3,1}(DQ_k) = 3k + 2$ and $\lambda_{3,1}(FDQ_k) = 2k + 3$. These results show that the structural differences between the two graph families lead to different labeling numbers. The explicit labeling constructions also demonstrate that the obtained lower bounds are sharp. For future research, the $L(3{,}1)$-labeling problem may be extended to other graph families constructed from multiple or interconnected cycles. It would also be interesting to investigate other distance-constrained labeling schemes, such as $L(h,k)$-labelings, on the double-quadrilateral windmill and flower graphs. Further studies may also consider related graph operations and generalized graph families derived from these structures.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest regarding the publication of this paper.

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